In work questions of roughness and bifurcations of dynamic systems, in particular the synergetic systems and chaos of various physical nature are considered. The basic provisions of the theory and a method of topological roughness developed by the author of work are presented. Use of the results received by the author are shown on examples of the known synergetic systems, such as Lorenz, Rössler, Belousov-Zhabotinsky's systems, Chua’s circuit, Henon’s map, "predator - prey", models of economic systems of Kaldor and Schumpeter, Rikitake's dynamo and also Hopf's bifurcation. The book is destined on a wide range of the researchers and scientists who are interested in synergetics and chaos of systems of various physical nature and also to students of the physical and mathematical and other natural-science and technical specialties dealing with problems of synergetics and dynamic systems.

Roman Omorov, doctor of engineering, professor, corresponding member of National Academy of Sciences of the Kyrgyz Republic (NAS KR), academician of the International engineering academy, FPI, LIPMO, area of researches: synergetics and control systems; sensitivity, roughness and bifurcations of systems; innovatics, intellectual property.

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# Theory of Topological Roughness of Systems:

# Applications to Synergetic Systems and Chaos

**CONTENTS**

## INTRODUCTION

Many parties of natural and technogenic processes and systems have identical or similar regularities of development and realization of the phenomena and events. The self-organization of their course and changes in time and space is uniting all these processes and the systems of various nature. Such processes and systems are a subject of researches of rather new science - synergetics which is engaged in the self-organized processes, the phenomena and the systems of the most various physical nature.

This science has gained broad development for the last decades and now interferes in many fields of science, since natural sciences - physics, chemistry, biology, geology – and finishing the inexact fields of sciences, such as economy, sociology, psychology, philosophy and also in the field of the equipment and technologies.

Many scientists set tasks not only researches of synergetic processes and systems now, but also managements of them for the purpose of achievement of their desirable development and dynamics. The research and control of chaotic behavior (chaos) in the systems of various physical nature are especially relevant in modern science.

The mathematical apparatus of synergetics as the directions of science is based first of all on the theory of dynamic systems and topology therefore the founder of the main ideas and methods of this new science should be considered the great French scientist Henri Poincare who has made an essential contribution to the theory of dynamic systems and has in essence based the section of mathematics - topology.

Further the huge contribution to development of the theory of dynamic systems was made by many scientific the 20th a century, including the Soviet scientists L.I. Mandelstam, A.A. Andronov, L.S. Pontryagin, V.I. Arnold and others. In development of the theory of stability of dynamic systems deposits of A.M. Lyapunov, S. Smale, R. Thom, V.I. Zubov and many other scientists are invaluable.

In modern statement the term of new science is synergetics is entered by the outstanding German physicist Hermann Haken in the late seventies of the twentieth century. The term "synergetics" comes from Greek "sinergen" is assistance, cooperation.

Self-organization in synergetic systems means spontaneous transition from the disordered state or even from chaos to ordered due to joint cooperative (synchronous) action of many subsystems or elements of systems.

The general sense of the synergetic ideas and methodology of synergetics consists in the following:

processes of destruction and creation, degradation and evolution in the nature and society have objective character;

creation processes (increase of complexity and orderliness) have a uniform algorithm, irrespective of the nature of systems.

Synergetic approach or paradigm of synergetics assumes that chaos and a disorder acts both as the destroyer and as the creator. The chaotic state comprises uncertainty – probability and accident. As a germ of self-organization serves "probability" - the orderliness arises through oscillations, stability through instability.

Necessary conditions of self-organization of systems and processes are:

the system – has to be open nonlinear and nonequilibrium;

the order in system arises through oscillations;

existence of positive feedback;

achievement by the system of the critical parameters of an order promoting and enhancing cooperative behavior of elements of system or subsystems.

Synergetic approach to systems and the phenomena of various nature introduces in science methodology absolutely new ideas not characteristic to classical traditional approach of science. These innovations of synergetics can be characterized by the following provisions:

generally, elaborate systems can't set a task of absolutely operated development, and it is only possible to set a task in a certain measure of the predicted self-governed development;

for difficult systems exists several alternative ways of development determined by the choice of a way behind a point (set) of branching (bifurcation);

the synergetics opens the new principles of superposition, assembly difficult of parts, creation of complex structures from simple when whole isn't equal to the sum of parts and qualitative other any more;

not the force of influence, but the correct topological structure or architecture of impact on system is defining in control of systems;

the synergetics discloses understanding why chaos can act as the creating beginning, the constructive mechanism of evolution as from chaos the new order can develop;

the synergetics discloses regularities and conditions of course of fast, avalanche processes and processes of nonlinear self-organized development.

At a research and control of synergetic systems questions of roughness and bifurcations are essential.

In classical statement questions of roughness and bifurcations of dynamic systems have been put at a dawn of formation of topology as new scientific direction of mathematics by the great French mathematician and the physicist H. Poincare, in particular, the term bifurcation is introduced by him for the first time and means literally "bifurcation" or otherwise from solutions of control of dynamic systems new decisions branch off. The roughness of dynamic systems at the same time is defined how properties of systems to keep a qualitative picture of splitting phase space into trajectories at small perturbation of topology, by consideration of relatives by the form of the equations of systems.

In modern terminology the word "bifurcation" is used as the name of any spasmodic change happening at smooth change of parameters in any system. Thus, bifurcation means the transition between spaces of rough systems happening through not rough areas (spaces).

Many fundamental results in the theory of roughness and bifurcations are received by A.A. Andronov and his school. In A.A. Andronov and L.S. Pontryagin's work (1937) the concept of roughness is for the first time given and qualitative criteria of roughness which in is called a concept of roughness according to Andronov-Pontryagin subsequently are formulated.

In the works of the author given in the list of references to the present book the results developing a concept of roughness according to Andronov - Pontryagin, the problems of roughness and bifurcations of dynamic systems allowing to investigate and solve quantitatively, in particular, effectively applied to synergetic systems are received.

Mathematical bases of the developed theory are based on a mathematical apparatus of synergetics, the theory of roughness and bifurcations of dynamic systems.

The mathematical apparatus of synergetics covers many fields of modern mathematics as objects of this science are objects of different fields of science. First of all, mathematical models of synergetic systems are represented by the nonlinear equations of different types. But at a research of the nonlinear equations various methods of decisions are used it is also linearization with data of tasks to the linear equations, it and use of matrix methods.

The linear and nonlinear equations are classified on ordinary differential and differential, on determined and stochastic, with constant and variable coefficients (stationary and non-stationary), concentrated and in private derivatives.

Questions of the theory of stability, the theory of matrixes and the theory of fractals are also important for researches of synergetic systems.

The considered theory of topological roughness is development of the theory of roughness and bifurcations of dynamic systems for researches of synergetic systems for the purpose of forecasting of bifurcations (accidents) and chaos and also controlling of them.

# CHAPTER 1. MATHEMATICAL FUNDAMENTALS OF SYNERGETICS AND THEORY OF TOPOLOGICAL ROUGHNESS OF SYSTEMS

## 1.1. Ordinary differential equations

We will consider the differential equations in a normal form:

$$\dot{x}_i = f_i(x_1, x_2, \dots x_n, t), \; i = \overline{1,n} \tag{1.1.1}$$

where $\dot{x}_i$ = d$x_i$/$dt$, $i = \overline{1,n}$ *is* the first derivative of the variables $x_i = x_i(t)$ depending on an independent variable *t*.

If to write down (1.1.1) in a vector-matrix form, then designating

$x = [x_1, x_2, \dots x_n]^T, F(x,t) = [f_1(x_1, x_2, \dots, x_n t), \dots f_n(x_1, \dots x_n, t)]^T$,

T – the sign of transposing,

we will have

$$\dot{x} = F(x,t)\ . \tag{1.1.2}$$

The solution of system (1.1.2) or (1.1.1) on an interval $\Delta = t_2 - t_1$ is called set of *n* of the functions $x_i = \xi_i(t)$, defined on an interval Δ such that substitution them in system (1.1.2) turns each equation of this system into identity on all interval Δ.

If a vector – function *F (x, t)* doesn't depend obviously on *t* time, i.e. the system of the differential equations (1.1.2) has an appearance

$$\dot{x} = F(x), \tag{1.1.3}$$

that this system of the equations is called autonomous (stationary).

Important task in the theory of the differential equations is the so-called task of Cauchy which is formulated as follows.

To find the solution of $x_i = \xi_i(t), \mathrm{i} = 1,2,\dots,n$ of system of the differential equations (1.1.1) defined on some interval Δ, $t_0$ containing a point, and meeting conditions

$$\xi_i(t_0) = x_{i0}, i = \overline{1,n}\,, \tag{1.1.4}$$

and $t_0$, the $x_{i0}$ in advance set numbers.

Values $t_0$, $x_{i0}$ $(i = \overline{1,n})$ are called initial values for the decision $\xi_i(t),\dots,\xi_n(t)$ and conditions (1.1.4) is entry conditions.

If to enter into consideration $(n + 1)$ measured Euclidean space with coordinates of $x_1, x_2, \dots, x_n, t$, then set of $n$ functions $x_i = \xi_i(t)$ will present to the line in this space, and initial values $x_{i0}, x_{10}, \dots, x_{n0}, t_0$ present a point to $(n+1)$ measured space (in Fig. 1.1, $n = 1$).

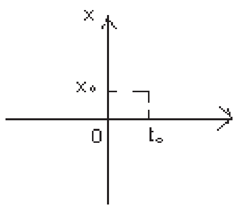



Fig. 1.1.

The most important of theorems at the solution of a task of Cauchy are theorems of existence and uniqueness of decisions from entry conditions and parameters.

We will formulate these theorems without reduction of proofs.

**Theorem 1.1.1.** For the system of the differential equations (1.1.2), if the functions $f_i(x_1, \dots x_n, t)$ are continuous on $t$ and meet Lipschitz's condition on $x_1, \dots x_n$ in some area G, i.e.

$$\left|f_i(x_1, x_2, \dots, x_{j1} \dots x_n, t) - f_i(x_1, \dots, x_{j2}, \dots x_n, t)\right| \leq \mathcal{L}\left|x_{j1} - x_{j2}\right|, \tag{1.1.5}$$

where $i = \overline{1, \text{n}}; j = \overline{n, 1}, \mathcal{L} = const$ Lipschitz's, then exists and besides the only solution of $x_i = \xi_i(t), \text{i} = \overline{1, \text{n}}$ systems (1.1.2), meeting entry conditions

$$x_i(t_0) = x_{i0}, i = \overline{1, \text{n}}, \tag{1.1.6}$$

defined on some piece Δ, $t_0$ containing a point.

**Theorem 1.1.2.** For the system of the differential equations (1.1.2), if the functions $f_i(x_1, \dots x_n, t)$ are continuous on *t* and in some area G, and $x = \xi(x_0, t_0, t)$ the solution of this system meeting entry conditions (1.1.6) and defined on a piece meet Lipschitz's condition on $|t-t_0| \leq \Delta$, for any $\varepsilon > 0$ exists it $\delta(\varepsilon, \Delta) > 0$, that other solution of $x = \tilde{\xi}(\widetilde{x_0}, t_0, t)$, meeting entry conditions

$$\tilde{\xi}(\widetilde{x_0}, t_0, t) = \widetilde{x_0},$$

where $||x_0 - \widetilde{x_0}|| < \delta$, it will be defined on the same piece Δ and satisfies to inequality

$$\| \xi(x_0, t_0, t) - \tilde{\xi}(\widetilde{x_0}, t_0, t) \| < \varepsilon, \tag{1.1.7}$$

where $\|\cdot\|$ - any vector norm.

True the similar theorem for parameters of system (1.1.2).

**Theorem 1.1.3.** For the system of the differential equations depending on parameters

$$\mu = [\mu_1, \dots \mu_m]^T : \dot{x} = F(x, \mu, t), \tag{1.1.8}$$

if $\xi(\mu_0, t)$ is the solution of system (1.1.8) at value of parameters $\mu = \mu_0$, satisfying to entry conditions: $\xi(\mu_0, t_0) = x_0$, for any $\varepsilon > 0$, it $\delta(\varepsilon, \Delta) > 0$, that if inequality $\| \mu - \mu_0 \| < \delta$, is fair that decision $\xi(\mu_0, t)$ it is defined on an interval exists $|t - t_0| < \Delta$, also satisfies to inequality

$$\| \xi(\mu, t) - \xi(\mu_0 t) \| < \varepsilon. \tag{1.1.9}$$

Theorems of continuous dependence of solutions of the differential equations on entry conditions and parameters are important very for practical models of various systems. Models of systems in practice decide on various errors therefore the above-stated theorems at small errors allow to operate with processes of real system authentically. Besides, the continuous dependence of decisions allows to receive continuous functional dependences of dynamics of systems on changes of entry conditions and parameters.

### 1.1.2. Linear ordinary differential equations

The linear systems of the differential equations in a normal form are presented in the form

$$\dot{x}_i = \sum_{k=1}^{n} a_{ik}(t)x_k + f_i(t), i = \overline{1,n}\ , \qquad (1.2.1)$$

or in vektor – a matrix form

$$\dot{x} = A(t)x + f(x), \qquad (1.2.2)$$

where $x = [x_1, \dots x_n,]^T$, $f(t) = [f_1(t),\dots,f_n(t)]^T$ are according to a vector of functions $x_i$, $f_i(t)$; A(t) is a matrix of coefficients of system:

$$A(t) = \begin{bmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn}(t) \end{bmatrix}. \qquad (1.2.3)$$

System (1.2.1.) or (1.2.2) is called uniform if $f_i(t) \equiv 0, \mathrm{i} = \overline{1,\mathrm{n}}$:

$$\dot{x} = A(t)x. \qquad (1.2.4)$$

The common decision of the equation (1.2.4) can be presented in the form

$$x(t) = x(0)e^{A(t)t}, \qquad (1.2.5)$$

where $x(0) = x_0$ is vector of initial values $x$ at $t_0 = 0$.

For the equations with constant coefficients $\dot{x} = Ax$, the decision will have an appearance

$$x(t) = x(0)e^{At}.$$

In case the matrix $A$ is brought to a diagonal look

$$A = M\Lambda M^{-1},$$

where $M$ is a reduction matrix to a diagonal view with columns own vectors of a matrix $A$, and $\Lambda = diag\{\lambda_i, \mathrm{i} = \overline{1, \mathrm{n}}\}$ is diagonal matrix, $\lambda_i$ is own values of a matrix $A$, then the solution of the differential equation will have an appearance

$$x(t) = x(0)Me^{\Lambda t}M^{-1}.$$

### 1.2. Linear differential equations

Functions which are defined only in some points of $t_1$, $t_2$, ..., i.e. at the discrete moments of a variable $t$ (as a rule, time) are called trellised. We will consider the trellised functions determined only in equidistant points of $t = nT$, where $n$ is any integer, in $T$ is the constant called by the discretization period. These functions are designated by $f(nT)$ (Fig. 1.2).

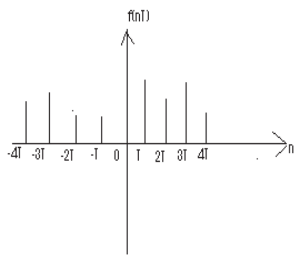



Fig. 1.2.

For brevity the $T$ falls further. Expression

$$\Delta f(n) = f(n+1) - f(n), \qquad (1.2.1)$$

is called the final difference of the first order of the trellised f(n) function, for brevity just the first difference. The first difference from trellised function $\Delta f(n)$ is called the difference of the second order of the trellised $f(n)$ function, etc. It is possible to consider the differences of any order of $i$ = 2,3, …, ($n$) designated $\Delta^2 f(n)$ , etc. For the difference of any order of $k$ the formula is fair

$$f(n) = \sum_{\upsilon=0}^{k} (-1)^{\upsilon} \binom{k}{\upsilon} f(n + k + \upsilon), \qquad (1.2.2)$$

where $\binom{k}{\upsilon} = C_{k}^{\upsilon} = \frac{k!}{\upsilon!(k-\upsilon)}$.

The final differences of trellised functions are functions, a discrete analog of derivatives in a continuous case.

At statement of the return problem of finding of the trellised *F (n)* function for which the Δ*F (n)* function is the first difference we have:

$$F(n) = \sum_{k=0}^{n-1} f\,(k). \qquad (n = 1,2, \dots) \qquad (1.2.3)$$

calls the *F(n)* function an antiderivative for trellised function *F(nT)* and summation in this case is an integration analog for continuous functions.

Now we will pass directly to consideration of the differential equations.

The ratio connecting trellised function *x (n)* and its differences to some order $k$:

$$\Phi\left(n, x(n), \Delta \times (n), \dots, \Delta^{k} \times (n)\right) = 0, \qquad (1.2.4)$$

is called the differential equation. Using (1.3.2) expression (1.3.4) it is possible to transform to a look

$$\Phi_1\big(n, x(n), x(n+1), \dots, x(n+k)\big) = 0. \qquad (1.2.5)$$

If the ratio (1.2.5) contains in an explicit form functions *x (n)* and *x(n+k),* then the differential equation (1.2.4) is called $k$ order equation. If in the course of reduction of the equation (1.2.4) to a look (1.2.5) functions *x (n)* can mutually be destroyed, at the same time the differential equation of a look turns out.

$$\Phi_2\,\big(n, x(n), x(n+2), \dots, x(n+k)\big) = 0.$$

(1.2.6)

Replacing *m=n+1*, we will receive

$$\phi_2(m-1), x(m), \dots, x(m+k-1)) = 0. \qquad (1.2.7)$$

The equation (1.2.7) is the differential equation about $k$-1.

Trellised function *x(n)* which turns the equation (1.2.4) or (1.2.5) into identity is called the solution of the differential equation.

We will consider the differential equation of an order of *k* resolved concerning function *x (n+ k):*

$$x(n+k) = F\big(h, x(n), \dots, x(n+k-1)\big). \qquad (1.2.8)$$

The linear differential equation of an order $k$ is called the equation

$$a_0(n)\Delta^r x(n) + a_1(n)\Delta^{r-1}x(n) + \cdots + a_{r-1}(n)\Delta x(n) + a_r(n)x\big(n = f(n)\big),$$
$$r \geq k, \qquad (1.2.9)$$

where *f(n)*,$a_0(n)$, $a_2(n), \dots a_r(n)$ are the set trellised functions.

Equation (1.2.9.) is called non-uniform if *f(n)*≠0 and uniform if *f(n)*=0. The equation (1.2.9) non-stationary if $a_i(n)$ depends on *n* and stationary if coefficients $a_i$ is constant. In that case, the uniform linear differential equation with constant coefficients given in a form (1.2.5) takes a form

$$\theta_0 x(n+k) + \theta_1 x(n+k-1) + \dots + \theta_{k-1} x(n-1) + \theta_k x(n) = 0, \qquad (1.2.10)$$

where $\theta_i = \sum_{v=0}^{i} (-1)^{i-v} \begin{pmatrix} k- & v \\ i- & v \end{pmatrix} a_v$, $\theta_k = 0, \quad \theta_0 \neq 0.$ (1.2.11)

The theorem is fair.

**Theorem 1.2.1.** If $n \geq n_0$ exists the fundamental system of decisions $\xi_1(n), \dots \xi_k(n)$ of the uniform differential equation

$$x(n+k) + \theta_1(n)x(n+k-1) + \dots + \theta_k(n)x(n) = 0, \qquad (1.2.12)$$

that common decision of this equation is expressed by a formuls

$$\xi(n) = \sum_{i=1}^{k} c_i \, \xi_i(n), \quad (n \geq n_0)$$

where $c_i (i = \overline{1,k})$ are any constants.

In case of constant coefficients we have

$$\xi(n) = \sum_{i=1}^{k} c_i \lambda_i^n, \tag{1.2.13}$$

where $\lambda_i, (i = \overline{1,k})$ are roots (characteristic numbers) of the characteristic equation of system (1.2.12).

$$\lambda^k + b_1 \lambda^{k-1} + \ldots + b_k = 0. \tag{1.2.14}$$

For the system of the differential equations with coefficients constants

$$x(n+1) = Ax(n), \tag{1.2.15}$$

where the $A$-matrix of coefficients, $A = [a_{ij}]$, $(i, j = \overline{1,k})$, the decision will have an appearance

$$x_i(n) = c_{ij} \lambda_j^n, \;\; i = \overline{1,k}. \tag{1.2.16}$$

where $\lambda_j$ are roots of the characteristic equation.

## 1.3. Nonlinear differential equations

The equation of a look (1.1.1), (1.1.2) is called the nonlinear differential equations if functions in the right part include nonlinear dependences on any $x_i$ or $t$ variables.

The solution of such equations in a general view is very difficult task of special types. Therefore in practice use qualitative and approximate methods of solutions of such equations. We will consider some of methods of the solution of the nonlinear differential equations.

**A.** *Method of consecutive approximations.*

By means of this method it is possible to receive the solution of a task of Cauchy for any differential equation (linear or nonlinear) or for the system of the differential equations satisfying with a condition of the theorem of existence and uniqueness of decisions.

We will consider the differential equation of the first order

$$\dot{x} = f(x, t). \qquad (1.3.1)$$

If решение $x = \xi(t)$, meets an entry condition

$$\xi(t_0) = x_0 \ , \qquad (1.3.2)$$

that we have the equivalent integrated equation.

$$\xi(t) = x_0 \ + \int_{t_0}^{t} f\,(\xi(\tau), \tau) d\tau. \qquad (1.3.3)$$

At the solution of the equation (1.3.1) by method of consecutive approximations as zero approach $\xi_0(t)$ any function is chosen (for example, $\xi_0(t) = x_0$ ) and it is substituted instead of $\xi(t)$ in the right part (1.3.3). The first approach the solution of the equation (1.3.1) in a look is found

$$\xi_1(t) = x_0 \int_{t_0}^{t} f\,(\xi_0(\tau), \tau) d\tau. \qquad (1.3.4)$$

Further in the right member of equation (1.4.3) instead of $\xi(t)$ function $\xi_1(t)$, determined by a formula (1.4.4) is substituted. As a result of integration the second approach of the decision $\xi_2(t)$, etc. n-e turns out approach will be defined by expression

$$\xi_n(t) = x_0 \int_{t_0}^{t} f\,(\xi_{n-1}(\tau), \tau) d\tau. \qquad (1.3.5)$$

It is proved that $n$ approach $\xi_n(t)$ at $n \to \infty$ will aspire to $\xi(t)$ - to the solution of the equation (1.3.1).

The same way it is possible to find the solution of $x = \xi\,(t)$ of system of the differential equations

$$\dot{x}_i = f_i(x_1, \dots, x_{n,}(t), \qquad i = \overline{1,n}, \tag{1.3.6}$$

meeting entry conditions

$$\xi_i(t_0) = x_{i0}, \;\; i = \overline{1,n}, \tag{1.3.7}$$

Despite simplicity of a method, it has a number of shortcomings which have limited its practical application.

**B.** From calculation of integrals the *method of broken lines of Euler* which also belongs to simple methods of approximate solutions of the differential equations is free.

We will find the solution of the equation of the first about (1.3.1), meeting entry conditions (1.3.2), considering that the equation (1.3.1) meets living conditions and uniqueness. So-called ε is the approximate solution of the equation (1.3.1) is found as follows.

On the plane *t, x* (Fig. 1.3) is under construction δ – network such that for set small ε > 0 at | $t$-$\tilde{t}$ | < δ and | $x$- $\tilde{x}$ | < δ correct inequality $|f(x,t) - f(\tilde{x},\tilde{t})| < \varepsilon$.

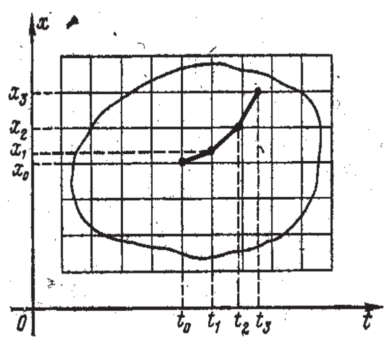



Fig. 1.3.

From a point ($x_0$ , $t_0$) draws a straight line

$$x - x_0 = f(x_0 - t_0)(t - t_0), \tag{1.3.8}$$

before crossing from one of the parties of the corresponding square. From a point of intersection ($x_1$ ,$t_1$ ) draws a straight line

$$x - x_1 = f(x_1, t_1)(t - t_1), \qquad (1.3.9)$$

and so on.

Thus, some broken line which approximates a required integrated curve turns out. Choosing the corresponding quantity of a step δ, it is possible to bring closer with any set degree of accuracy the constructed broken line to the exact decision.

**B.** *The solution of the equations by means of power series.*

In case right parts of system of the equations

$$\dot{x}_i = f_i(x_1, \dots, x_{n,}(t), \qquad i = \overline{1, n}, \qquad (1.3.10)$$

we will spread out in power series in the neighborhood of a point $(x_{10}, \dots x_{n0}, t_0)$, the solution of system (1.3.10) can be found in the form of power series with uncertain coefficients. Equating coefficients at the corresponding degrees of $t$ it is possible to calculate coefficients of these ranks.

**C.** *Method of decrease in an order (reduction method).*

We will consider the common decision of system of the equations

$$\dot{x} = f_i\left(x_1, \dots, x_{n,}(t)\right), \qquad i = \overline{1, n} \qquad (1.3.11)$$

in some area G:

$$x_i = \varphi_i(c_1, \dots, c_{n,}t), \; i = \overline{1, n} \qquad (1.3.12)$$

from which by the choice of any constant $c_1, \dots, c_n$ can receive any decisions belonging to area G.

Resolving a ratio (1.3.12) rather any constant $c_1, \dots, c_n$ we will receive $n$ of the equations

$$\begin{array}{c} \psi_1(x_1, \dots, x_{n,}t) = c_1 \\ \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \\ \psi_n(x_1, \dots, x_{n,}t) = c_n \end{array} \qquad (1.3.13)$$

The set of equalities (1.3.13) is called the general integral of system (1.3.11), and each of these equalities is the first integral of this system.

The first integral can be defined also as the ratio containing in the left part independent variable both required functions and accepting constant value if instead of required functions to substitute any solution of system (1.3.11).

If it is known any the first integral, then an order of system can be lowered on unit. Really, let $\psi(x_1, \dots, x_{n,} t) = c$ is first integral systems (1.3.11). Then, expressing from him one of the unknown $x_k$ functions through $t$, other unknown functions $x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n$ and any constant c:

$x_k = \psi(x_1, \dots, x_{k-1}, x_{k+1}, \dots x_{n,} t, c)$, and substituting this expression instead of $x_k$ in the initial system of the equations, we receive system from $n$-1 of the equations with $n$-1 unknown functions.

Thus, the order of system of the equations is lowered on unit. Similarly, if $r$ of independent first integrals are known, then the order of system goes down on $r$ of units, thereby strongly simplifying the solution of initial system of the equations (1.3.11).

**D.** *Method of harmonious linearization.*

This method is applied to approximate determination of parameters of the periodic solution of the nonlinear differential equation. By means of a method of harmonious linearization it is possible to find out existence of periodic solutions of the nonlinear differential equations and also to determine parameters of these decisions and to investigate its stability.

We will consider a method of harmonious linearization on the example of the autonomous system of the first order of a look

$$x_2 = f(x_1, \dot{x_1}), \qquad (1.3.14)$$

where $x_1(t) = a \sin\omega t$ is harmonious function with an amplitude $a$ and frequency ω.

Further the equation (1.3.14) can be written down

$$x_2 = f(asin\omega t, a\omega cos\omega t). \quad (1.3.15)$$

Decomposing *x₂(t)* function in a row of Fourier:

$$x_2 = a_0/2 + \sum_{k=1}^{\infty}(a_k cosk\omega t + b_k sink\omega t), \quad (1.3.16)$$

through the sequence of transformations we will receive harmoniously linearized equation of the nonlinear equation (1.3.14) in the following look:

$$x_2 = q(a, w)x_1 + \frac{q'(a,w)}{w}\dot{x_1}, \quad (1.3.17)$$

where $q(a, \omega)$ and $q'(a, \omega)$ are coefficients of harmonious linearization have an appearance:

$$q(a, \omega) = \frac{1}{\pi a}\int_0^{2\pi} f(asin\omega t, a\omega cos\omega t)sin\omega t d(\omega x), \quad (1.3.18)$$

$$q'(a, \omega) = \frac{1}{\pi a}\int_0^{2\pi} f(asin\omega t, a\omega cos\omega t)cos\omega t d(\omega t). \quad (1.3.19)$$

## 1.4. Stochastic nonlinear differential equations

As it has been shown in the previous chapter by consideration of any real system, process, the corresponding variables can be divided on microscopic and macroscopic, and depending on objectives to consider those processes which adequately reflect the required level of the solution. Obviously, depending on consideration level temporary scales will be various. So, for example, microscopic processes are played in much smaller time scales in comparison with macroscopic processes and fluctuations the reflecting processes in separate parts of system, in microscopic processes happen in significantly in short time scales, than in macroscopic processes.

The stochastic differential equations describe macroscopic processes with fluctuations in a certain mathematical statement.

First of all, we will consider the basic concepts of the theory of casual processes.

### 1.4.1. Casual processes and their main statistical characteristics

Stochastic function is called function which value at each value of an independent variable is a random variable. Stochastic functions for which an independent variable is t time are called casual processes or stochastic processes.

Any $x_i\ (t)$ function which is equal to casual process $x\ (t)$ as it to result of experience, is called realization of casual process. It is in advance impossible to predict on what realization stochastic process will go.

For any fixed time point, for example, $t=t_1$, realization of casual process of $x_i(t)$, $i = 1, \dots, n$ represents concrete quantity, the value of stochastic function $x(t_1)$ is the random variable called by the section of casual process in time point of $t_1$ (Fig. 1.4),

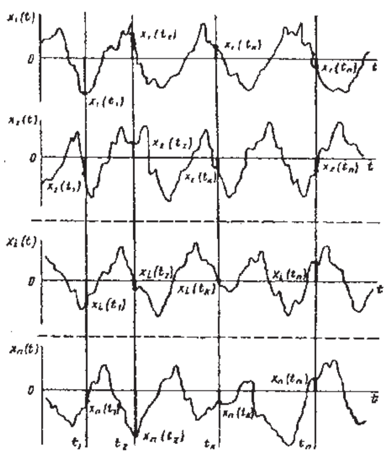


Fig. 1.4.

Statistical methods study not each of the realization of $x_i(t)$ forming multitude $x(t)$, and property of all set in general by means of averaging of properties, entering him realization.

Statistical properties of a random variable $x$ determine by her function of distribution (the integrated law of distribution) of $F(x)$ or density of probability (the differential law of distribution) of $w(x)$.

Random variables can have various laws of distribution: uniform, normal (Gauss), exponential (Poisson), etc.

At the normal law of distribution or Gaussian distribution random variable $x$ completely is defined by mathematical mean (average value) $m_x$ and average quadratic deviation $\sigma_x$.

Analytical expression of function of distribution in this case

$$F(x) = \frac{1}{\sqrt{2\pi 6_x}} \int_{-\infty}^{x} e^{-(x-m_x)^2} / 2\, 6^2 x dx, \qquad (1.4.1)$$

respectively density of probability is defined by a formula

$$w(x) = \frac{1}{6_x\sqrt{2\pi}} e^{-\frac{(x-m_x)^2}{26^2}}. \qquad (1.4.2)$$

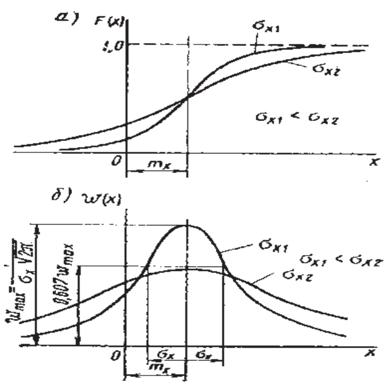



Fig. 1.5.

Function graphs of distribution of $F(x)$ and density of probability of $w_x$ for various values $\sigma_x$ are shown in Fig. 1.5.

For casual process there is also a concept of function of distribution of $F(x, t)$ and density of probability of $w(x, t)$ which depend on the fixed time point of $t$ and from some chosen $x$, i.e. are functions of two variables $x$ and $t$.

If x ($t_1$) the section of casual process in time point of t that one-dimensional function of distribution (the first order) of casual process $x(t)$ is called probability that the current value of casual process $x(t_1)$ in timepoint of $t_1$ doesn't exceed some set level (number) $x_1$, i.e.

$$F_1(x_1, t_1) = P\{x(t_1) \leq x_1\}. \tag{1.4.3}$$

If the $F_1$ function ($x_1$, $t_1$) has a private derivative on $x_1$, i.e.

$$w_1(x_1, t_1) = \frac{\partial F(x_1, t_1)}{\partial x_1}. \tag{1.4.4}$$

That the $w_1$ function ($x_1$, $t_1$) is called the one-dimensional density of probability (the first order) of casual process.

Quantity

$$w_1(x_1, t_1)dx_1 = P\{x_1 < x(t_1) \leq x_1 + dx_1\}, \tag{1.4.5}$$

represents probability that $x(t)$ to be in time point of $t=t_1$ in an interval and $x_1$ to $x_1+dx_1$.

Functions $F_1(x, t)$ and $w_1(x,t)$ are the simplest statistical characteristics of casual process. They characterize casual process separately in his separate sections without opening an interconnection between sections of casual process, i.e. between mutual values of casual process in various time points.

Probability that $x(t)$ will be no more $x_1$ at $t=t_1$ no more $x_2$ at $t=t_2$, i.e.

$$F_2(x_1, t_1; x_2, t_2) = P\{x(t_1) \leq x_1; x(t_2) \leq x_2\}, \tag{1.4.6}$$

function is called two-dimensional function of distribution (the second order), and

$$w_2(x_1, t_1; x_2, t_2) = \frac{\partial^2 F_2(x_1,t_1;x_2,t_2)}{\partial x_1 \partial x_2}, \quad (1.4.7)$$

is called the two-dimensional density of probability (the second order).

Quantity

$$w_2(x_1, t_1; x_2, t_2)dx_1 dx_2 = P\{x_1 < xt_1 \leq x_1 dx_1; \quad x_2 < x(t_2 \leq x_2 +)dx_2\}, \quad (1.4.8)$$

it is equal to probability that $x(t)$ at $t=t_1$ will be in the range from $x_1$ to $x_1+dx_1$, and at $t=t_2$ in the range from $x_2$ to $x_2 + dx_2$.

Also $n$ is measured functions of distribution and density of probability which are applied very seldom are similarly entered.

Among a set of casual processes have special remarkable characteristics the special class of the casual processes called Markov, casual to processes by name the great Russian mathematician A.A. Markov who for the first time has studied and which have put the main ideas of use of such processes.

For Markov casual processes, knowledge of value of process at the time of $t(k)$ already comprises all information on future course of process what can only be taken from behavior of process up to this point. In case of Markov casual process for definition of probabilistic characteristics of process in time point of $t_m$ it is enough to know probabilistic characteristics for any one previous time point $t_k$. Knowledge of probabilistic characteristics of process for other previous values of time, for example $t_e$, doesn't add information necessary for find $x(t_m)$.

For Markov process fairly

$$(x_1, t_1; x_2, t_2; \ldots x_{n,} t_n) =$$
$$= \frac{w_2(x_1,t_1;x_2,t_2) w_2(x_2,t_2;x_3,t_3) \ldots w_2(x_{n-1},t_{n-1};x_2,t_n}{w_1(x_1,t_1) w_1(x_2,t_2) \ldots w_1(x_{n-1},t_{n-1})}, \quad (1.4.9)$$

i.e. all density of probability of Markov process are defined from the two-dimensional density of probability. Thus, Markov casual processes are completely characterized by the two-dimensional density of probability.

In practice except functions distribution and density of probability use rather simpler more often, though less total characteristics of casual processes similar numerical the characteristics of casual processes similar to numerical characteristics of random variables. Treat such characteristics: population mean, dispersion, mean square of casual process, correlation function, spiral density and others.

Mathematical mean (average value) of $m_x(t)$ of casual process $x(t)$ call quantity

$$m_x(t) = M\{x(t)\} = \int_{-\infty}^{\infty} x\, w_1(x,t)dx, \qquad (1.4.10)$$

where $w_1(x,t)$ density of probability of casual process $x(t)$ of the first order.

The mathematical mean of casual process $x\ (t)$ represents some nonrandom (regular) function of time $m_x(t)$ about which are grouped and concerning which all realization of this casual process (Fig. 1.6) fluctuates.

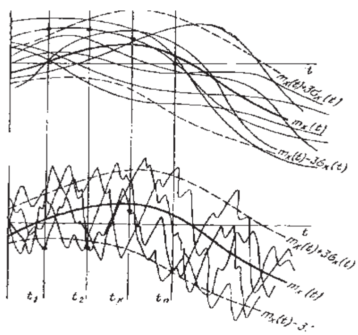



Fig. 1.6.

Thus, the population mean is average value of casual process on a set (to averages on ensemble, statistical averages) as it represents probabilistic and average value of an infinite set of realization of casual process.

Mean square of casual process is called quantity

$$\overline{x^2}(t) = M[\{x(t)\}^2] = \int_{-\infty}^{\infty} x^2 w_1(x,t)dx. \qquad (1.4.11)$$

The central casual process $x(t)$ is a deviation of casual process $x(t)$ from its mathematical mean $m_x(t)$:

$$\dot{x}(t) = x(t) - m_x(t). \qquad (1.4.12)$$

Dispersion of casual process, is equal to population mean of a square of the aligned casual process:

$$D_x(t) = M[\{\dot{x}(t)\}^2] = \int_{-\infty}^{\infty} \{x - m_x(t)\}^2 \, w_1(x,t)dx. \qquad (1.4.13)$$

The quantity $\overline{x^2}(t), D_x(t)$ and $m_x(t)$ is connected by a ratio

$$\overline{x^2}(t) = D_x(t) + m_x^2 \, . \qquad (1.4.14)$$

In practice often use the following statistical characteristics of casual process

$$\bar{x}(t) = \sqrt{\overline{x^2}(t)} = \sqrt{D_x(t) + m_x(t)}, \qquad (1.4.15)$$

Average quadratic deviation of casual process

$$6_x(t) = \sqrt{D_x(t)}. \qquad (1.4.16)$$

The mathematical mean and dispersion are important characteristics of casual process, but they don't give a sufficient idea of what character will be had by separate realization of casual process.

For the characteristic of internal structure of casual process, i.e. for accounting of communications between values of casual process in various time points or, otherwise, for accounting of degree of variability of casual process, the concept about correlation (auto correlated) function of casual process is entered.

Correlation function of casual process *x(t)* call nonrandom function of two arguments $R_x(t_1; t_2)$ which for each couple randomly of the chosen values of arguments of $t_1$ and $t_2$ is equal to population mean of the work of two random variables and the corresponding sections of casual process:

$$R_x(t_1 t_2) = M[\dot{x}(t_1)\, \dot{x}(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{x_1 - m_x(t)\} \bullet \{x_2 - m_x(t_2)\}\, \omega_2(x_1,t_1;x_2 t_2)dx_1 dx_2,$$

(1.4.17)

where $w_2(x_1,t_1;x_2,t_2)$ is density of probability of the second order; $(x-m_x(t))$ - the aligned casual process; $M_x(t)$ is mathematical mean of casual process.

Casual processes share on: stationary and non-stationary depending on changes of statistical characteristics.

Stationary in narrow sense call casual process $x(t)$ if his $n$-dimensional functions of probability at any $n$ don't depend on shift of all points of $t_1, t_2, \dots, t_n$ along time axis on identical quantity τ, i.e.

$$F_n(x_1,t_1;x_2t_2;\dots;x_nt_n) = F_n(x_1,t_1+\tau;\ x_2t_2+\tau;\dots\ x_nt_n+\tau);$$

$$w_n(x_1,t_1;x_2t_2;\dots;x_nt_n) = w_n(x_1,t_1+\tau;\ x_2t_2+\tau;\dots\ x_nt_n+\tau). \qquad (1.4.18)$$

Thus, here two processes $x(t)$ and $x(t+\tau)$ have identical statistical properties for any τ, i.e. statistical characteristics of casual process are invariable in time.

Stationary in a broad sense call casual process $x(t)$ which population mean is constant:

$$M[x(t)] = m_x = \text{const}, \qquad (1.4.19)$$

and correlation function depends only on one variable is the difference of arguments $\tau = t_1 - t_2$:

$$R_x(\tau) = R_x(t_1,t_1+\tau) = M[x(t_1), x(t_1+\tau)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{x_1 - m_x(t_1)\}\bullet\{x_2 - m_x(t_1+\tau)x$$

$$w_2(x_1,x_2,\tau)\,dx_1dx_2. \qquad (1.4.20)$$

For normal casual processes a concept of stationarity of broad and narrow sense coincide. Except average value on a set $x(t) = m_x(t)$ exists average value on time of $t$ which is defined on the basis of observation of separate realization of casual process $x(t)$ throughout a sufficient long time of $T$.

$$\tilde{x} = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}x(t)dt, \qquad (1.4.21)$$

if this limit exists.

Generally average value on a set $\bar{x}$ and average value on time $\tilde{x}$ are various. But these quantity are identical to so-called ergodic casual processes:

$$\bar{x} = \tilde{x}. \qquad (1.4.22)$$

For ergodic processes of expression for many statistical characteristics considerably become simpler:

$$R_x(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \{x(t) - \tilde{x}\}\{x(t+\tau) - \tilde{x}\}dt, \qquad (1.4.23)$$

$$D_x = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \{x(t) - \tilde{x}\}\{x(t) - \tilde{x}\}dt, \qquad (1.4.24)$$

$$D_x = R_x(0) = \text{const}, \qquad (1.4.25)$$

$$\sigma_x = \sqrt{D_x} = const. \qquad (1.4.26)$$

For characteristics of interrelation of two casual processes of $x(t)$ and $g(t)$ the mutual correlation $Rxg$ functions are entered:

$$R_{xg}(t_1 t_2) = M[x(t_1), g(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{x_1 - m_x(t_1)\}\{g - m_g(t_2)\}\ \omega_2(x_1, t_1; g, t_2) dx_1 d_g. \qquad (1.4.27)$$

For ergodic casual processes:

$$R_{xg}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \{x(t) - \tilde{x}\}\{g(t+\tau) - \tilde{g}\}dt, \qquad (1.4.28)$$

where $x(t)$ and $g(t)$ are any realization of stationary casual processes $x(t)$ and $g(t)$.

One more characteristic of casual processes is spectral density $s_x(\omega)$ is defined by the following formula:

$$s_x(\omega) = \int_{-\infty}^{\infty} R_x(\tau) l^{-j\omega\tau} d\tau, \qquad (1.4.29)$$

or $$s_x(\omega) = \int_{-\infty}^{\infty} R_x(\tau)\cos\omega\tau d\tau - j\int_{-\infty}^{\infty} R_x(\tau) S \sin\omega\tau d\tau = 2\int_{0}^{\infty} R_x(\tau)\cos\omega\tau d\tau, \qquad (1.4.30)$$

respectively

$$R_x(\tau)= \frac{1}{\Pi}\int_0^\infty S_x(\omega)\cos\omega\tau d\omega, \tag{1.4.31}$$

$$D_x= R_x\,(0)=\frac{1}{\Pi}\int_0^\infty S_x(\omega)d\omega, \tag{1.4.32}$$

The mutual spectral density of $s_{xg}$ (ω) of two casual processes *x(t)* and *g(t)* is determined by a formula:

$$S_{xg}{}^{j\omega} = \int_{-\infty}^{\infty} R_{xg}(\tau)l^{-j\omega\tau}d\tau\,, \tag{1.4.33}$$

respectively

$$R_{xg}\,(\tau)= \frac{1}{2\Pi}\int_{-\infty}^{\infty} S_{xg}(j\omega)l^{j\omega\tau}d\omega, \tag{1.4.34}$$

In conclusion of this subsection we will provide some data of distribution of Poisson which often meets by consideration of synergetic systems.

The sequence of the events which are taking place in the casual time points which are continuously distributed on a numerical axis are called a stream of events. Streams of the events answering to the following conditions are called Poisson:

1) if ($t_1, t_2$) and ($t_3, t_4$) are any not blocked time intervals, then probability of emergence of any number of events during one of them doesn't depend on that how many events appear during another;

2) the probability of emergence of one event during an infinitesimal interval of time ($t, t + \Delta t$) is the infinitesimal quantity of an order $\Delta t$;

3) the probability of emergence more than one event during time interval (t, t + Δ t) is infinitesimal the highest order in comparison with Δt.

The law of distribution of Poisson determined by a formula is characteristic of a Poisson stream of events of *X*

$$P_m = \frac{\mu^m}{m} \bullet l^{-\mu},\ m = 0,1,2\ldots, \qquad (1.4.35)$$

where $\mu = M[X]$ is mathematical mean of number of events in the range of $\Delta t = t - t_0$.

Quantity $\nu(t) = d\mu / dt$ (1.4.36) is called the average density or intensity of a stream of events. Now in this section we will pass the stochastic differential equations to consideration of some important synergetic systems in models.

### 1.4.2. Stochastic differential equations

Stochastic equation of a look

$$dx_l(t) = B_l(x(t))dt + \sum_m g_{lm}(x(t))d\upsilon_m(t), \qquad (1.4.36)$$

at

$$d\upsilon_m = 0, \qquad (1.4.37)$$

$$\overline{\vartheta_m(t)d\vartheta_\iota(t)} = \delta_{lm}dt, \qquad (1.4.38)$$

where $x=x(t)$ is a vector of conditions of system (a vector of coordinates) with the $x_i(t)$ elements; $B_i$ is an element of a vector of the compelling forces; $g_{im}$ is function of dependence of amplitude of the fluctuating forces $d\vartheta_m$ on a vector of states $x(t)$; $\delta_{im}$ is Kronecker's symbol, is called Ito's equation.

In the simplified look Ito's equation takes a form

$$\dot{\bar{x}}_l(t) = \bar{B}_l(x(t)). \qquad (1.4.39)$$

Ito (1.4.36) equation is connected with widely known stochastic equation of Fokker – Planck:

$$\partial F(x)/\partial t = -\sum \frac{\partial}{\partial_\kappa}[B_\kappa(x)F(x)] + \frac{Q}{2}\Sigma \frac{\partial^2}{\partial x_k \partial x_l}[\sum_m g_{km}g_{lm}F(x), \qquad (1.4.40)$$

where $F(x)$ is function of distribution of probabilities, Q is measure of quantity of fluctuations.

Also Stratanovich's equation which turns out by consideration of processes in the middle of an interval is connected with Ito's equation $[t_{i-1}, t_i]$:

$$x_l(t) = x_l(t_0) = \int_{t_0}^{t} B_l(x)dt^1 = \int_{t_0}^{t} \bar{g}_{lm}(x)d\upsilon_m(t). \tag{1.4.41}$$

In essence Stratanovich's equation turns out from Ito's equation when replacing:
$\bar{g}_{lm} = g_{lm}$,

$$\bar{B}_l = B_l - \frac{1}{2}\frac{\partial g_{em}}{\partial x_r} g_{km}. \tag{1.4.42}$$

Special case of the equation of Ito is Lanzheven's equation:

$$Dx_l(t) = B_l(x(t)dt. \tag{1.4.43}$$

## 1.5. Theory of matrixes

### 1.5.1. Vectors

In the beginning we will consider a concept about vectors and operations over them. A vector of dimension of $n$ is called a set $n$ of the elements $x_i$ ,
$i=\overline{1,n}$ presented in the column form

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}. \tag{1.5.1}$$

If number $x_i$ is vector elements, we have a numerical vector $x$ if $x_i$ is variable, we have a vector of variables if $x_i = f_i$ are functions, $x$ is a vector function, etc.

If to enter operation of transposing on replacement of a column with line and to designate the sign of transposing is $T$ (sometimes), then a vector (1.5.1.) it is possible to write down in a lower case look

$$x^T = [x_1, x_2, ..., x_n]^T = [x_1, x_2, ... x_n]^1 = x^1 \ . \qquad (1.5.2)$$

Over vectors the following binary operations – addition (subtraction), multiplication are carried out.

The sum of two vectors *x* and *y* with the $x_i$ and $y_i$ , $i = \overline{1, n}$ , registers $x + y$ and is defined by a vector

$$x{+}y = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix} . \qquad (1.5.3)$$

The difference differs only in replacement of the sign (+) on (-).

Multiplication by a scalar *c*, i.e. on a vector with one element, is communicative operation.

$$cx = xc = \begin{bmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_n \end{bmatrix} . \qquad (1.5.4)$$

The scalar product of a vector on a vector is a scalar, and is carried out on a formula

$$(x,y) = \sum_{i=1}^{n} x_i y_i \ . \qquad (1.5.5)$$

The scalar product of vectors has the following properties:

- commutativity: $(x, y) = (y, x)$; (1.5.6 a)

- distributivity:

$$(x+y, z+\omega) = (x1,z) + (x, \omega) + (y,z) + (y, \omega); \qquad (1.5.6 б)$$

- associativity: $(cx, y) = c\,(x, y)$, $(x, (y, z)) = (x, y)$. (1.5.6 c)

The scalar product $(x, x)$ is a square of "length" of a material vector

$$(x,x) = \sum_{i=1}^{n} x_i^2 = |x|. \qquad (1.5.7)$$

The vector work of vectors *x*, at is designated [*x, s*] or $x\times y$ is also equal to *z* vector to it that length of a vector of *z* is equal

$$|z|=|x\times y|=|x|\cdot|y|\cdot Sin\varphi, \qquad (1.5.8)$$

where $\varphi$ is a corner between vectors *x* and *y*, the same time vector *z, x, y* form the so-called right three (Fig. 1.7), the vector of *z* is perpendicular to the plane of vectors *x*, *y*.

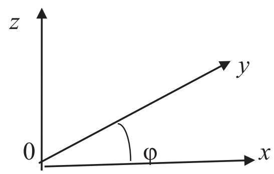



Fig. 1.7.

The vector work of vectors has the following properties:

commutativity with the return sign:

$$x\times y=-(x,y)\ ; \qquad (1.5.9\ a)$$

associativity: $(cx)\times y=c(x\times y)$ ; (1.5.9 b)

distributivity: x×(y+z) =x×y+x×z. (1.5.9 c)

Two material vectors (with material elements) are called orthogonal if the ratio (*x, y*) = 0 is carried out.

Important concept for vectors is the norm of vectors which characterizes the size (length) of a vector. In *n* dimensional valid space the concept about norm of a vector, it that to each vector of $x\in R$ some real non-negative number of $\|x\|$ is put in compliance, so is entered that for any vectors *x*, *y* from $R^n$ and any scalar the following conditions are satisfied:

1. $\|x + y\| \leq \|x\|+\|y\|$,

2. $||x|| = |c|\ ||x||$, (1.5.11)

3. $||x|| > 0$, if $x \neq 0$.

There are three types of the vector norms answering to conditions (1.5.11):

- "cubic" norm

$$||x|| = \max |x_i|, \quad i = \overline{1,n}; \tag{1.5.12}$$

- "Hermite" or "spherical" norm

$$||x||_2 = \sqrt{\sum_{i=1}^{n} | x_i |^{\bar{2}}}\ ; \tag{1.5.13}$$

- "octahedral" norm

$$|| x||_3 = \sum_{i=1}^{n} | x_i |\ . \tag{1.5.14}$$

### 1.5.2. Matrixes

We will pass to consideration of matrixes.

Table of the elements $a_{ij}$, i = $\overline{1,n}$; j=$\overline{1,m}$ in shape

$$A = \begin{bmatrix} a_{11}\ a_{12}...a_{1m} \\ a_{21}\ a_{22}...a_{2m} \\ ------ \\ a_{n1}\ a_{n2}...a_{nm} \end{bmatrix}, \tag{1.5.15}$$

is called a matrix of dimension of *nxm*. If *n=m*, then we have a square matrix.

Generally *A* can be elements of a matrix as real, complex numbers, and variables or functions, respectively a matrix *A* will be called a material (complex) numerical matrix, a matrix of variables or functions.

In case of a square matrix of *n* is called a matrix order.

Sometimes at designation of a matrix use the reduced designation

$$A = [\, a_{ij} \,], \ \mathrm{i} = \overline{1,n}; \quad \mathrm{j} = \overline{1,m}. \tag{1.5.16}$$

Two matrixes are equal each other, in only case when, when all corresponding elements are equal, i.e. matrixes $A$ yes $B$ are equal if $A=B$, $a_{ij} = в_{ij}$, $\forall$ (for all): $i,j = i = \overline{1,n}$; $j = \overline{1,m}$.

*Operations over matrixes*:

Addition of matrixes:

$$A + B = [a_{ij} + b_{ij}]; \tag{1.5.17}$$

The work of a matrix on a scalar:

$$cA = Ac = [ca_{ij}]; \tag{1.5.18}$$

multiplication of matrixes:

$$A \bullet B = [a_{ij}]\,[b_{kl}] = [c_{ij}] = C. \tag{1.5.19}$$

It is obvious that the number of columns $m$ of a matrix $A$ has to coincide with number of lines of a matrix of $B$. In case quantities of $n$ of lines of a matrix $A$, coincides with number of columns of a matrix $B$, then we will have a square matrix of $C$. Generally a matrix $C$ rectangular, dimensions of $nxm$.

Multiplication of a matrix $A$ by a vector $x$ is carried out on a formula

$$C = Ax = [a_{ij}]\,[x_i] = [c_i], \tag{1.5.20}$$

where the $C$ is vector with the elements $C_i = \sum_j^n a_{ij} x_i$, $i = \overline{1,n}$.

Properties of operations over matrixes:

not commutativity (generally):

$$AB \neq BA; \tag{1.5.21a}$$

associativity:

$$(AB)C = A(BC); \tag{1.5.21b}$$

distributivity:

$$A(B + C)=AB+AC. \quad (1.5.21c)$$

Transposing of matrixes is operation of replacement of lines with columns or on the contrary, i.e.

$$A^T = [a_{ij}]^T = [a_{ji}]. \quad (1.5.22)$$

We will consider some special types of matrixes.

1. Symmetric matrixes are such matrixes which satisfy to ratios

$$A = A^T \text{ or } (A^1), \quad (1.5.23a)$$

i.e. for such matrixes

$$a_{ij} = a_{ji} . \quad (1.5.23b)$$

Symmetric matrixes belong to material matrixes with material elements.

If a matrix $A = [a_{ij}]$ with complex numerical elements, then symmetric matrixes call Hermite matrixes.

2. It is called Hermite matrixes the matrixes satisfying to a ratio

$$A = \overline{A}^T , \quad (1.5.24)$$

where $\overline{A}$ is a complex – interfaced to $A$ a matrix, i.e. if elements $A$, $a_{ij} = \alpha_{ij} \pm j\beta_{ij}$,

that elements $\overline{A}$: $\overline{a}_{ij} = \alpha_{ij} \mp j\beta_{ij}$. These matrixes are called by name the great French mathematician Charles Hermite.

Sometimes instead of $A$=$A^T$ write down simply A*.

Orthogonal matrixes of $T$ are such matrixes for which

$$T^T T = I , \quad (1.5.25)$$

where $I$ is a single matrix at which is on unit diagonal and other elements are equal to zero.

3.Unitary matrixes are an analog orthogonal in case of complex matrixes, i.e. for them

$$T^*T = I\,. \tag{1.5.26}$$

The diagonal matrix is such matrix which has on diagonal any numbers (at least one is other than zero), and other elements are equal to zero. Are designated diagonal, $\Lambda$ or $diag\{a_i\}$,

$$\Lambda = diag\{a_{ii}\}, i = \overline{1,n}\,, \tag{1.5.27}$$

where $a_{ii}$ are diagonal elements.

5. Quazydiagonal from in a complex diagonal matrix it is a matrix the closest to diagonal material representation of a complex matrix, i.e.

$$\Lambda_q = diag_q\{\alpha_i\beta_i\}. \tag{1.5.28}$$

on diagonals blocks 2x2 with elements $\begin{bmatrix} a_j & \beta_j \\ -\beta_i & a_i \end{bmatrix}$,

which correspond to diagonal elements of a diagonal matrix with complex elements $\Lambda = diag\{\alpha_i \pm j\beta\}$, and other elements $\Lambda_q$ are equal to zero.

6. Positive attributive and negative certain matrixes.

Properties of positive definiteness or negatively definiteness of matrixes are very important properties for practical applications.

If matrix $A = [a_{ij}]$ is a material symmetric matrix and the square form of $n$ order

$$Q_n(x) = \sum_{i,j=1}^{n} a_{ij} x_i y_j > 0, \tag{1.5.29}$$

for all uncommon $x_i$, $x_j$, that $Q_n(x)$ is called positively certain form, and a matrix of $A = [a_{ij}]$ is positively certain matrix. Similarly, if $H$ Hermite matrix and form

$$P_n(x) = \sum_{i,j=1}^{n} h_{ij} x_i \bar{x}_j > 0, \tag{1.5.30}$$

for all complex uncommon $x_i$, $x_j$, the $P_n(x)$ form and a matrix of $H$ is called positively certain.

If inequalities (1.5.29) and (1.5.30) have the return signs, then the corresponding forms and matrixes are called negatively certain.

If signs in (1.5.29) and (1.5.30) inequalities mild i.e. a look $\geq 0$, then square forms and matrixes are called positively semi-certain.

The square $Q_n(x)$ and $P_n(x)$ forms can be written down in a look

$$Q_n(x) = x^T Ax, \tag{1.5.31a}$$

$$P_n(x) = x^T Hx. \tag{1.5.31b}$$

7. Jordan form of matrixes.

Initial Jordan form of a matrix $A$ is called $J$ which is made of quasidiagonal blocks with diagonal elements in the form of own values of a matrix $A$ taking into account their frequency rate and naddiagonalny or subdiagonal units in blocks:

$$J = [J_1 , J_2 , \ldots J_k], \tag{1.5.32}$$

For example, if $\lambda_1$, $\lambda_2$, $\lambda_3$ own values $A$, frequency rates respectively $2,3,1$, then we have

$$J = \begin{bmatrix} \begin{matrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{matrix} & 0 & 0 \\ 0 & \begin{matrix} \lambda_2 & 1 & 0 \\ & \lambda_2 & 1 \\ & & \lambda_2 \end{matrix} & 0 \\ & 0 & \lambda_3 \end{bmatrix} . \tag{1.5.33}$$

The Jordan form characterizes the maximum reducibility of matrixes $A$ to a quasidiagonal look by linear transformations.

We will consider concepts about norms of matrixes.

Norm of a matrix $A$ generally is called the non-negative real number designated by $\|A\|$, it that

$$\|A\| = \sup\frac{\|Ax\|}{\|x\|}, \quad x\in R x\neq 0, \qquad (1.5.34)$$

where sup is the upper bound of a set; $\|x\|$ and $\|Ax\|$ any vector norms.

In that case, we have the following types of norms of the matrixes subordinated to the relevant standards of vectors:

at "cubic" norms of vectors:

$$\| A \|_1 = \max \sum_{k=1}^{n} | a_{ik} |; \qquad (1.5.35a)$$

at "octahedral" norms of vectors:

$$\| A \|_3 = \max_{1 \le k \le n} \sum_{i=1}^{m} | a_{ik} |; \qquad (1.5.35b)$$

at "Hermite" (or spherical) norms of vectors:

$$\| A \|_2 = \sqrt{\rho}, \qquad (1.5.35\ c)$$

where ρ - the maximum characteristic number of a matrix of $AA^*$ (or $A^TA$), *is called spectral norm;*

at axiomatic introduction of the matrix norm coordinated with vector norm ||·|| we have two more appearance of norms

$$\| A \|_E = \sqrt{\sum_{i,k} | a_i k |^2}, \text{ Euclidean norm} \qquad (1.5.35\ d)$$

and $$\| A \|_n = n \max_{ij} |a_{ij}|; \qquad (1.5.35\ e)$$

at different types of norms vektors $\|Ax\|$ and $\|x\|$ in (1.5.32) we have one more norm of matrixes

$$\| A \|_n = \max_{1\le,k\le n} |a_{ik}|. \qquad (1.5.35\ f)$$

Norms $\|A\|_n$, $\|A\|_E$ and $\|A\|_2$, which satisfies to the following ratios are often used

$$\frac{1}{n}\|A\|_n \le \|A\|_2 \le \|A\|_n\ , \qquad (1.5.36\ a)$$

$$\frac{1}{\sqrt{n}}\|A\|_E \le \|A\|_2 \le \|A\|_E\ . \qquad (1.5.36\ b)$$

Besides of transposing of matrixes over matrixes are carried out operation of the address of matrixes when find $A^{-1}$ matrix the return to a matrix $A$, such that

$$AA^{-1} = E = I = diag\{1\}, \qquad (1.5.37)$$

i.e. multiplication of the return matrix by a matrix $A$ gives a single matrix.

The return matrix exists if the determinant of a matrix $A$ isn't equal to zero, i.e. the matrix $A$ isn't degenerated $|A|\neq 0$.

If $det\ A=0$, in this case a matrix $A$ is degenerated, and the return matrix of $A^{-1}$ doesn't exist.

In such cases and also when the matrix $A$ not square, rectangular dimensions of $nxm$ is entered into consideration a so-called pseudo-return matrix the designated $A^+$, which always exists.

Properties of the pseudo-return matrix:

1) the matrix of $A^+$ has dimension of $nxm$ if $A$ dimensions of $nxm$;
2) the space of columns of a matrix of $A^+$ coincides with space of lines of a matrix $A$ and vice versa;
3) the pseudo-return to $A^+$ is a matrix $A$: $(A^+)^+ = A$;
4) generally $AA^+ \neq I$, but $AA^+ = P$, where $P$ is matrix the carrying-out design on space of columns of a matrix $A$ yes satisfying $A\overline{x}=Pb$; $\overline{x}$ is an optimal solution of not joint equation of $Ax=b$.

We will consider characteristics of matrixes.

The main characteristics of matrixes are:

the *determinant* designated or d*et* *A* or |*A*|;

*trace*, tr (*A*) or sp (*A*)

*norm*, ‖*A*‖;

*rank* of a matrix, Rank(A)

*eigenvalue* λ(*A*); and *eigenvectors* of matrixes;

*spectral radius* ρ;

*singular numbers* σ(*A*);

*conditionality number* C{*A*}.

We will consider these characteristics of matrixes:

Over matrixes it is possible to carry out linear transformations of similarity, so, that if there is some nondegenerate matrix of *M* of the same order, as *A*, then the transformed matrix

$$B=M^{-1}AM, \qquad (1.5.38)$$

has the same own values, as a matrix *A*.

1. *Matrix determinant*.

The determinant of a numerical matrix is the real number characterizing some properties of a matrix.

For example, not triviality of determinant of the matrix designated or |*A*|, or det(*A*), guarantees that the return matrix of $A^{-1}$ exists.

The determinant of a matrix *A* is equal to parallelepiped volume in *n* - measured space, constraction on the vectors coinciding with lines of a matrix *A* (Fig. 1.8).

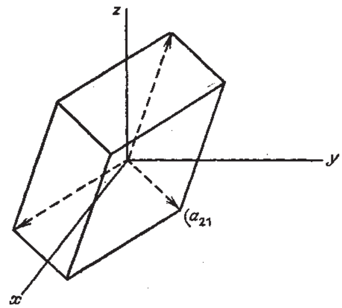



Fig. 1.8.

The most general way of calculation of determinants it through decomposition on algebraic additions, any line or any column, for example, through algebraic addition i-line will have an appearance:

$$det\, A = ai1\, Ai1 + ai2\, Ai2 + \cdots + ain\, Ain, \qquad (1.5.39)$$

where algebraic addition $A_{ij}$ is $M$ submatrix determinant $ij$, the taken sign

$$Aij = (-1)^{i+j}\, det\, Mij\,. \qquad (1.5.40)$$

The submatrix of $M_{ij}$ is formed by deletion of $i$ of a line and $j$ of a column of a matrix $A$.

It is the scheme of calculation of determinant of a matrix $n$ of an order, allows step by step reducing an order of matrixes to an elementary order 2, to calculate as much as high order determinant.

Determinants are used in many problems of linear algebra, in particular for calculation of the return matrix and the solution of system of the algebraic equations.

Calculation of $A^{-1}$ through det$A$ is carried out on a formula

$$A^{-1} = \frac{adj\,A}{\det A}\ , \qquad (1.5.41)$$

where adj $A$ is algebraic addition of a matrix $A$, calculated on (1.5.40).

The solution of the $Ax=b$ system is found through $\det A$ on the following expression

$$x = A^{-1}b\,\frac{(adj\ A)b}{\det A}, \qquad (1.5.42)$$

or by Kramer's rule: $j$ an element of a vector $x = A^{-1}b$ is equal to $x_j = \frac{\det \beta_j}{\det A}$, where

$$\beta_j = \begin{bmatrix} a_{11} & a_{12}\cdots & b_1 \cdots a_{1n} \\ \vdots & \vdots & \vdots \\ a_{n1}\ldots & a_{nz}\ldots & b_n \ldots a_{nn} \end{bmatrix}, \qquad (1.5.43)$$

in a matrix $A$ replace $j$ a column with $b$ vector, receive a matrix $\beta_j$.

2. We will consider a concept about a matrix *trace*.

The sum of diagonal elements of a matrix $A$ is called a trace of a matrix $A$ yes Sρ $A$ or tτ $A$ is designated.

$$\text{S}\rho A = \textstyle\sum_i a_{ii}, \quad i = \overline{1,n}. \qquad (1.5.44)$$

The trace of a matrix $A$ is connected with own values of a matrix $A$ a ratio

$$S\rho A = \textstyle\sum_i \lambda_i, \quad i = \overline{1,n}. \qquad (1.5.45)$$

Properties of a trace of a matrix:

distributivity:

$$\text{S}\rho(A+B)=\text{S}\rho A+\text{S}\rho B, \qquad (1.5.46a)$$

commutativity: $$\text{S}\rho(AB)=\text{S}\rho(BA). \qquad (1.5.46b)$$

3.Matrix *rank.*

Rank of a matrix $A$ is called the size rank ($A$) or r($A$) equal to the greatest order of minors of a matrix, other than zero, $A$. Minors are determinants of all submatrixes of $M_{ij}$ of a matrix $A$, i.e.

$$rank(A) = r(A) = \max_{i,j} \det M_{i,j}. \quad (1.5.47)$$

Important property of a rank of matrixes is the following property:

if the $T$ matrix nondegenerate, then the matrix rank $TA$ is equal to a rank of a matrix $A$, i.e.

$$r(TA) = r(A) \quad \det T \neq 0. \quad (1.5.48)$$

The rank of a square form $(x, Ax) = x^T Ax$ coincides with a rank of a matrix $A$, i.e.

$$r = (x, Ax) = x^T Ax, \quad (1.5.49)$$

so the rank of square forms ($x$, $Ax$) remains at the nondegenerate $x=Ty$ transformations.

4. The concept about *norms* of matrixes is earlier considered by us.

5. *Eigenvalues* and *eigenvectors* of matrixes.

Eigenvalues and eigenvectors of matrixes are one of the most important characteristics of the matrixes used in many applied problems of science.

The number λ is called eigenvalue of a matrix $A$ with the corresponding zero eigenvector $x$ if it satisfies to the equation

$$\det(A - \lambda I) = 0, \quad (1.5.50)$$

which is the characteristic equation for a matrix $A$.

The matrix of an order of $n$ has $n$ eigenvalues $\lambda_i$, $i = 1, \dots, n$ satisfying to the characteristic equation (1.5.50).

Eigenvector $x$ matrixes $A$ satisfy to the algebraic equation

$$Ax=\lambda x. \quad (1.5.51)$$

Properties of eigenvalues of a matrix $A$.

1. $\sum_i \lambda_i = \sum_i a_{ii} = S\rho A, \quad (1.5.52a)$

$$2.\ \prod_i \lambda_i = \lambda_1\lambda_2 \dots \lambda_n = \ detA. \qquad (1.5.52b)$$

3. If a matrix $A$ is triangular, then eigenvalues $\lambda_1,\dots, \lambda_n$ in accuracy coincide with the diagonal elements $a_{11}, a_{22},\dots, a_{nn}$ of a matrix $A$.

4. Eigenvalues of a matrix $A^2=AA$ are equal $\lambda_1^2$, $\lambda_2^2,\dots, \lambda_n^2$ and each eigenvector of a matrix $A$ are eigenvector of a matrix of $A^2$:

$$A^2x = A\lambda x = \lambda Ax = \lambda^2 x.$$

5. Linear dependence and independence of eigenvectors of a matrix $A$.

If the matrix $A$ has no multiple eigenvalues that $n$ of eigenvectors are linearly independent, i.e.

$$\Sigma c_i x_i = c_1x_1 + \dots + c_nx_n = 0 \Leftrightarrow c_1 = \dots = c_n = 0,$$

where $c_i, i = \overline{1,n}$ any constants.

If for vectors $\{x_i, i = \overline{1,n}\}$ expression (1.5.53) is carried out in case of any $c_i \neq 0$, vectors of $x_i$ linearly are dependent.

In case the matrix $A$ the $nxn$ size has $n$ of linearly independent eigenvectors, then when transforming similarity with a matrix of $M$ made of eigenvectors of a matrix $A$ as columns, the transformed matrix of $M^{-1}Ax$ is a diagonal matrix $\Lambda$ at which on diagonal there are eigenvalues of a matrix $A$

$$M^{-1}Ax = \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & \ddots & \lambda_n \end{bmatrix}. \qquad (1.5.54)$$

As not all matrixes have $n$ are linearly independent eigenvectors, not all matrixes of a diagonalability.

In case of lack of $n$ linearly independent eigenvectors of a matrix $A$, i.e. not all eigenvalues are various, the closest form to which it is possible to transform like a matrix $A$ is the Jordan form of a matrix of $J$ (see (1.5.33)).

6. *Spectral radius of matrixes.*

Spectral radius of a matrix $A$ is called size ρ, equal to the maximum module of eigenvalues of a matrix $A$.

$$\rho(A) = \max_i |\lambda_i|. \qquad (1.5.55)$$

Spectral radius ρ is used as the size of norm of matrixes $\|A\|_2$, when it is calculated ρ matrixes of $A^T A$ or $A^H A^{H*}$.

Spectral radius is often used for assessment of convergence of various iterative procedures, for example, at convergence assessment the procedure of consistently approximate solution of system of the algebraic equations

$$Ax=b, \qquad (1.5.56)$$

by method of splitting of a matrix $A$, when condition of convergence of the decision $x_k \to x$ :

$$\rho(A_1^{-1}) = \max_i |\lambda_i (A^{-1} A_2)| < 1, \qquad (1.5.57)$$

where matrix $A_1$, $A_2$ are such that $A = A_1^{-1} A_2$.

Moreover, speed of convergence of iterations

$$A_1 e_{k+1} = A_2 e_k, \ e_k = x - x_k \ , \qquad (1.5.58)$$

depends from $\rho(A^{-1} A_2)$.

7. *Singular numbers of matrixes*.

Any matrix $A$ dimension of $nxm$ can be presented as the work of three special matrixes:

$$A = Q_1 \Sigma Q_2^T, \qquad (1.5.59)$$

where the $Q_1$ is orthogonal matrix of the $m$x$m$ size, the $Q_2$ is orthogonal matrix of the $n$x$n$ size, and a matrix $\Sigma$ the $m$x$n$ size also has the special diagonal form:

$$\Sigma = \begin{bmatrix} \sigma_1 & & & & \\ & \sigma_2 & & & \\ & & \ddots & 0 & \\ & & & \sigma_z & \\ 0 & & & & \end{bmatrix}, \qquad (1.5.60)$$

where $\sigma_i > 1$, $i = \overline{1,\tau}$, $\tau$ is rank of a matrix $A$.

Representation (1.5.59) is called singular decomposition of a matrix $A$, and numbers $\sigma_i$, $i = \overline{1,\tau}$ is called singular numbers of a matrix $A$.

Using singular decomposition (1.5.59), the pseudo-return matrix can be calculated on a formula

$$A^+ = Q_2 \Sigma^+ Q_1^T, \qquad (1.5.61)$$

where $\Sigma^+$ is a matrix at which nonzero diagonal elements are equal $6_i^{-1}$, $i = \overline{1,r}$ and other zero; equality for orthogonal matrixes $Q_2^T = Q_2^{-1} = Q_2^+$, too and for $Q_1$ is used.

Squares of singular numbers $\sigma_i$ is eigenvalues of a matrix of $A^T A$ therefore the maximum singular number $\sigma_{max} = \sigma_i$ is norm of a matrix $\|A\|_2$, i.e.

$$6_{\max} = \sqrt{\rho} = \|A\|_2. \qquad (1.5.62)$$

8. *Number of conditionality of matrixes.*

The number of conditionality of matrixes receives more and more application in various applied problems of science where various matrix methods and computing algorithms are used.

The number of conditionality is one of important multiplicative parameters of matrixes.

Number of conditionality of a matrix $A$ is called quantity

$$c\{A\} = \| A \| \| A^{-1} \|, \qquad (1.5.63)$$

where $\|\cdot\|$ is any norm of a matrix.

Conditionality numbers c$\{A\}$ have the following properties:

1) $c\{A\} = c\{A^{-1}\}$; (1.5.64 a)

2) $c\{AB\} \leq c\{A\}\, c\{B\}$; (1.5.64 b)

3) $c\{A\} \geq \| I \| \geq 1$ ; (1.5.64 c)

4) $c\{k A\} = k^2 c\{A\}$ . (1.5.64 d)

Using various norms of matrixes it is possible to enter different types of numbers of conditionality of matrixes:

1) at "cubic" norms of matrixes:

$$C_1\{A\} = \| A \|_1 \, \|A^{-1}\|_1, \quad (1.5.65a)$$

2) at "octahedral" norms of matrixes:

$$C_3\{A\} = \| A \|_3 \, \|A^{-1}\|_3, \quad (1.5.65b)$$

3) at "Hermite" (at "spherical") norms:

1.$C_2\{A\} = \| A \|_2 \, \|A^{-1}\|_2,$ (1.5.65c)

2.$C_E\{A\} = \| A \|_E \, \|A^{-1}\|_E,$ (1.5.65d)

3.$C_n\{A\} = \| A \|_n \, \|A^{-1}\|_n,$ (1.5.65e)

4.$C_4\{A\} = \| A \|_4 \, \| A^{-1}\|_4.$ (1.5.65f)

According to norms of matrixes, from the listed types of numbers of conditionality often use numbers of $C_n\{A\}$, $C_E\{A\}$, $C_2\{A\}$, and also sometimes number

$$C_5\{A\} = \frac{\max|\lambda_i|}{\min|\lambda_i|} . \quad (1.5.65g)$$

For symmetric matrixes of $C_5\{A\}$ and $C_2\{A\}$ coincide.

Numbers of conditionality of different types, satisfy to the following ratios:

$$\frac{1}{n}C_n\{A\} \leq C_E\{A\} \leq C_n\{A\} \leq n\, C_E\{A\}\,, \qquad (1.5.66a)$$

$$C_2\{A\} \leq C_E\{A\} \leq n\, C_2\{A\}\,, \qquad (1.5.66b)$$

$$C_5\{A\} \leq C_2\{A\}\,.$$

(1.5.66c)

Thus, the smallest of the numbers of conditionality given here is, but from coordinated with norms a vector and matrixes the smallest is the number $C_2\{A\}$ which is called spectral number of conditionality of matrixes and has received the greatest practical application. Further if isn't stipulated we offer such number of conditionality of matrixes and for convenience we will lower the index 2, i.e. $C\{A\}$.

So, conditionality number

$$C_2\{A\}= C\{A\}=\frac{6\, max}{6\, min}=\sqrt{\frac{\lambda_{max}\,(A^+A)}{\lambda_{min}(A^+A)}}\,. \qquad (1.5.67)$$

Conditionality numbers $C\{A\}$ characterize proximity of matrixes to expressiveness i.e. as far as det$A$ is close to zero.

Broad application of numbers of conditionality is connected also with that circumstance that errors of calculations and solutions of the equations, at is inexact preset values of matrixes of coefficients or free constants, depends on numbers of conditionality of matrixes.

So for a problem of $Ax=b$, at errors in elements of matrixes $A$ or a vector of $b$ of an error of decisions are connected with initial errors the following ratios:

$$\frac{||\delta x\,||}{||x||} \leq C\{A\}\frac{||\delta b||}{||b||}, \qquad (1.5.68a)$$

$$\frac{||\delta x\,||}{||\,x+\delta x||} \leq C\{A\}\frac{||\delta A||}{||A||}. \qquad (1.5.68b)$$

### 1.5.3. Matrix equations

We will consider some matrix equations which are often used in tasks with use of methods of a matrix formalism.

A. *Matrix equation of Lyapunov.*

At the solution of questions of stability by a so-called second or direct method of Lyapunov, very often the task comes down to definition of positively certain symmetric matrix $P$ satisfying to the following matrix equation, called by the equation like Lyapunov:

$$A^T P + PA = -Q ,$$

where $Q$ is also some, the set symmetric positively certain matrix, in that specific case, believe $Q=I$ and the equation is considered

$$A^T P + PA = -I. \quad (1.5.69)$$

The system for which stability is defined has an appearance

$$\dot{x} = Ax , x(0)=x_0 , \quad (1.5.70)$$

where $x \in R^n$ – a vector of states.

In that case, if there is a matrix $P$ satisfying (1.5.70), then the system (1.5.71) is steady asymptotically.

B. *Matrix equation of Rikkati.*

This equation is uniform of the basic equations in a problem of the optimum equation of square criterion of quality.

If to consider current trends in the theory of control, in the general statement of tasks various short changings and the systems of various nature, the modern approaches to the theory of control based on the synergetic principles of self-organization of systems are required.

These tendencies lead to origin of the new theory of control is the synergetic theory of control which component is the synergetic theory of optimum control.

There are two types of the matrix equations Rikkati:

1) differential matrix Rikkati's equations:

$$\dot{P} = -PA - A^T P + PBR^{-1} B^T P - Q, \qquad (1.5.71)$$

where $R$ is required symmetric $n$x$n$ matrix; $A$, $B$ are the set matrixes; $R$ is positive and certain matrix; $Q$ is non-negative and certain matrix; all matrixes in (1.5.71) either stationary or time-dependent;

2) algebraic matrix equation of Rikkati:

$$A^T\bar{P} + \bar{P}A - \bar{P}BR^{-1} B^T\bar{P} = -Q, \qquad (1.5.72)$$

where $\bar{P}$ is required constantly positive and certain matrix of the $n$x$n$ size; other matrixes same, as for the differential matrix equation, but constants.

The equation (1.5.72) and (1.5.73) generally rather difficult, allowing analytical decisions only in the simplest cases.

The main way of the solution of the matrix equations like Rikkati this use of numerical methods of the solution of the matrix equations with use of modern computers.

C. *Matrix equation of Sylvester.*

This matrix equation is the main equation at the solution of tasks of modal control when the set range of the closed system is provided.

Consideration of this equation is also caused by requirements of the synergetic theory of control about which it was noted above.

So the matrix equation of Sylvester has an appearance

$$MG - AM = -BH, \qquad (1.5.74)$$

where $M$ is required $n$x$n$ a matrix with constant coefficients; $A$, $B$ are the set constant matrixes of dimensions according to $n$x$n$ and $n$x$r$; $H$ is $r$x$n$ a constant matrix; $G$ is a diagonal (quasidiagonal) matrix of an order of $n$; couples of matrixes ($A$, $B$) is form the controled couple; couple ($G, H$) is observed couple, i.e.

$$U=[B: AB: \ldots : A^{n-1}B],\ r(\mathcal{U}) = r\ (A), \qquad (1.5.75a)$$

$$N=[H ⋮ H\Gamma: \ldots : H\Gamma^{n-1}]^{T},\ r(N) = r(\Gamma). \qquad (1.5.75b)$$

D. *Matrix equation of similarity.*

This matrix equations necessary for search of a matrix of eigenvectors.

It is known that transformation of similarity with the $M$ matrix from columns in the form of own vectors of any matrix $A$ leads to a diagonal matrix $G = diag\ \{\lambda_{i,}\ i = \overline{1,n}\}$, or in case of the complex interfaced eigenvalues $\lambda_{i,}$ to quasidiagonal form with blocks $\begin{bmatrix}\alpha_i & \beta_i\\ -\beta_i & \alpha_i\end{bmatrix}$, i.e.

$$G = M^{-1}A\ M, \quad M\,G = A\ M. \qquad (2.5.76)$$

The equation (2.5.76) is matrix the similarity equations. This equation is a special case of the matrix equation in Frobenius's task of search of all matrixes of the $X$ permutable with this matrix $A$. Methods of the decision both analytical, and numerical for the last task are well-known from literature.

## 1.6. Theory of stability

Stability is one of the most important properties of any real system and expresses ability of systems reverts to the original state after removal of the enclosed perturbations.

### 1.6.1. Stability of linear systems

Consideration of questions of stability of systems is carried out on the basis of their mathematical models.

**A.** *Stability of the systems described by the linear ordinary differential equations.*

We will consider linear differential the look equations

$$\dot{x} = A(t)x + f(t), \tag{1.6.1}$$

where $\dot{x} = x(t) \in R^n$ is vector of states; *A(t)* is *n*x*n* a matrix of coefficients; *f(t)* is a vector – functions time-dependent *t*; $\dot{x} = dx/dt$ is first derivative.

If *f(t)*≡0, then system (1.6.1) is autonomous system, otherwise nonautonomous system. If *A(t)* is doesn't depend on time, then we have system stationary.

We will consider system autonomous stationary, i.e. let the system have an appearance

$$\dot{x} = Ax. \tag{1.6.2}$$

It is known that the system (1.6.2) is stability if all eigenvalues of a matrix *A* have not positive material parts, and elementary dividers corresponding to values with a zero material part simple.

Thus, if eigenvalues which are roots of the characteristic equation are known

$$\det(A - \lambda I) = 0, \tag{1.6.3}$$

that definition of stability of system (1.6.1) doesn't represent complexity, namely stability of system (1.6.2) requires also enough that

$$Re\lambda_i \leq 0, \quad i = \overline{1,n}\,, \tag{1.6.4}$$

at the same time, the system will be stability also at the multiple roots lying on an imaginary axis, but there is enough that they had simple elementary dividers, i.e. the corresponding cage in an initial Jordan form of a matrix *A* consisted of one element.

The system (1.6.1) asymptotically is stability if it is carried out

$$Re\lambda_i < 0, \quad i = \overline{1,n}\,. \tag{1.6.5}$$

The nonautonomous system (1.6.1) is also steady at stability of autonomous system (1.6.2).

In case of non-stationary systems with periodic coefficients, i.e. *A(t)* is a matrix with periodic elements, system

$$\dot{x} = A(t)x, \tag{1.6.6}$$

has the solution of a look

$$x\,(t) = B(t)e^{ct}, \tag{1.6.7}$$

where *B(t)* is a matrix with elements of the same period, as *A(t); C* is some constant matrix.

Thus, in this case stability of system is defined by eigenvalues of a matrix of *C*.

Without calculation of eigenvalues apply various criteria of stability which are divided into algebraic and frequency (graphic) and also special methods with use of the computer to definition of stability of linear systems.

**B.** *Stability of systems described linear differential to the equations. (stability of discrete systems).*

The linear differential equations describe so-called discrete systems, and also the systems constructed on discrete maps.

In a general view the differential equations register:

$$x(n+1)=F(n,x(n)), \tag{1.6.8}$$

where *x(n+1)* is a vector with the $x_i(n+1)$, $i = 1 \dots k$; and *F(n, x(n))* is a vector of the $F_i\big(n, x_1(n), x_2(n) \dots, x_k(n)\big)$.

If to enter Euclidean (Hermite) norm of a vector *x(n)*:

$$\| x(n) \|_E = \sqrt{\sum_{i=1}^{k} x_i^2 (n)}. \qquad (1.6.9)$$

that stability across Lyapunov of any decision ξ(*n*) the equations (1.6.8), under entry conditions ξ($n_0$) is formulated as follows.

The decision ξ(*n*) the equations (1.6.8) is called stability if for any ε > 0, there is it δ > 0, depending from ε and on $n_0$ that any decision φ(*n*) for which at $n = n_0$ true inequality

$$||\varphi(n_o) - \xi(n_o)|| < \delta, \qquad (1.6.10)$$

which satisfies at all values of a discrete argument $n \geq n_0$ to a condition

$$||\varphi(n_o) - \xi(n)|| < \varepsilon, \qquad (1.6.11)$$

The decision ξ(*n*) the differential equation (1.6.8) is called asymptotically stability if it is stability, and besides, there is such number H > 0, that from a condition $||\varphi(n_o) - \xi(n_o)|| < \mathrm{H}$, follows

$$\lim_{n - \infty} ||\varphi(n) - \xi(n)|| = 0. \qquad (1.6.12)$$

We will consider the linear systems described by the differential equations

$$x_i = \sum_j^k a_{ij} (n) x_j(n) + f_i(n), \ i = \overline{1,k}, \qquad (1.6.13)$$

or in a matrix and vector look

$$x(n+1)=A(n)x(n)+f(n), \qquad (1.6.14)$$

where *A(n)* is a matrix with the elements $a_{ij}(n), i = \overline{1,k}, j = \overline{1,k}$ ; *f(n)* is a vector with the elements $f_i(n), i = \overline{1,k}$ .

The system (1.6.14) nonautonomous non-stationary differential equations is stability if the system is stability corresponding autonomous (uniform):

$$x(n+1)=A(n)x(n). \qquad (1.6.15)$$

This non-stationary system is stability when all her decisions are limited and asymptotically is stability when all her decisions tend to zero at $n \to \infty$.

We will consider autonomous system with constant coefficients, i.e. stationary linear discrete system:

$$x(n+1)=Ax(n), \qquad (1.6.16)$$

where $A = [a_{ij}]$. $(j, i = \overline{1, k})$ is a matrix of constant coefficients.

Stability of system (1.6.16) is defined by two conditions:

1. all eigenvalues $\lambda_i$ a matrix $A$ on the module don't exceed unit,

$$|\lambda_i| \leq 1; \qquad (1.6.17)$$

2. to eigenvalues which modules are equal to unit there correspond simple elementary dividers of a matrix $A$ (a Jordan form with one cage of the corresponding eigenvalue).

The system (1.6.16) is stability asymptotically if all eigenvalues of a matrix $A$, $\lambda_i$, $i = \overline{1, k}$ on the module there is less unit, i.e.

$$|\lambda_i| < 1, \; i = \overline{1, k}. \qquad (1.6.18)$$

In this case there are also various criteria (algebraic, frequency, computing (on the computer)) which allow to define stability of discrete systems without calculation of eigenvalues of a matrix $A$ or without solving the corresponding characteristic equation.

We will consider only algebraic criteria Schur-Kohn and discrete option of criterion Gurvits.

The criterion Schur - Kohn consists in the following.

Let the characteristic polynom for (1.6.16) have an appearance

$$det(A - \lambda I) = D(\lambda) = в_0\lambda^n + в_1\lambda^{n-1} + \ldots в_{k-1}\lambda + в_k. \qquad (1.6.19)$$

We will consider a material case when all coefficients of a polynom of $D(\lambda)$ are material.

Then determinants Schur-Kohn are calculated:

$$\Delta M = \begin{vmatrix} B_{0m} & B_{km} \\ B_{km}^T & B_{0m}^T \end{vmatrix}, \tag{1.6.20}$$

where $B_{0m}$, $B_{km}$ are matrixes:

$$B_{0m} = \begin{bmatrix} b_0 & 0 \dots & 0 & \dots 0 \\ b_1 & b_0 \dots & 0 \dots & 0 \\ b_{m-1} & b_{m-2} \dots & b_{m-3} & \ddots b_0 \end{bmatrix};$$

$$B_{km} = \begin{bmatrix} b_k & b_{k-1} \dots & b_{k-2} \dots b_{k-m+1} \\ 0 & b_k \dots & b_{k-1} \dots b_{k-m+2} \\ 0 & 0 & 0 \ddots b_k \end{bmatrix},$$

i.e. $\Delta_m = \left|B_{om}B_{om}^T - B_{km}^T B_{km}\right|$, (1.6.21)

where $m=\overline{1,k}$.

Determinants of $\Delta_m$ have 2*m* lines and 2*m* columns therefore the labor input of their calculations increases at increase in degree of *k*.

Calculation of determinants of Gurvits will require much the smaller number of computing operations (Gurvits's determinants have *m* lines and *m* columns).

The discrete option of criterion Gurvits consists in the following.

Conformal mapping of the left half-plane of roots of the characteristic equation in a single circle is used (Fig.1.9).

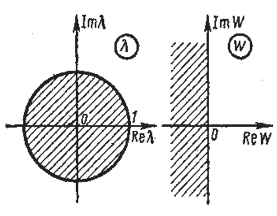



Fig. 1.9.

Such mapping is carried out by fractional and linear transformation:

$$x=\frac{1+w}{1-w}\,. \qquad (1.6.21)$$

Substituting in (1.6.20) we receive some polynom.

$$D_1(w)=a_0w^k+a_1w^{k-1}+\ldots+a_{k-1}w+a_k, \qquad (1.6.22)$$

where $a_i=f(d_i),\ i=0,k,\ \ j=o,k.$

Thus, zero polynom of $D(x)$ lying in a single circle on the plane $x$ are displayed in zero polynom of $D_1(w)$ lying in the left half-plane of variable $w$.

Further for establishment of stability of system (1.6.16) Gurvits's criterion concerning system with a characteristic polynom of $D_1(w)$ is fair, the i.e is formed Gurvits's matrix but coefficient of $a_i,\ i=0,k,$ and determinants as in a continuous case are calculated.

### 1.6.2. Stability of nonlinear systems

For definition of stability of nonlinear systems there are also various methods, both algebraic, and frequency or computing.

**A.** *Second (direct) method of Lyapunov.*

The greatest application from methods of a research of stability was received by the second or direct method of Lyapunov. This method allows to define stability of nonlinear systems, without solving the equations.

We will consider the equation

$$\dot{x}=f(x), \qquad (1.6.23)$$

where $x\epsilon R^r;\ F(x)$ is the nonlinear smooth vector function having continuous private derivatives on all arguments in some area $G$: $\|x\| \le c$ = const.

Stability of the trivial decision (1.6.23) $x(t) \equiv 0$, is defined by the following conditions:

the system is stability if:

1) there is positively certain function $V(x)$;

2) a derivative of function $V(x)$ owing to the equation (1.6.23)

$W(x) = dv/(dt)$ is negative sign, i.e.

$V(x) > 0, \quad W(x) \leq 0.$

For asymptotic stability of a trivial condition of system:

$$V(x) > 0, \qquad W(x) < 0. \tag{1.6.24}$$

The instability of a trivial condition of $x \equiv 0$ of system (1.6.23) can be established on the following conditions:

1) there is any continuous function $V(x)$ meeting a condition of $V(0) = 0$;

2) the derivative of function $V(x)$ owing to system (1.6.23) $W(x)$ is definite sign (positively or negatively);

3) in any vicinity of the beginning of coordinates ($x$=0) there are points in which the sign of function $V(x)$ coincides with the sign $W(x) = \partial v/\partial t$.

**B.** *Definition of stability of systems on the first approach.*

Stability of the trivial decision (1.6.23) can be determined by the equations of the first approach when the right part (1.6.23) is linearized in the neighborhood of $x = 0$. Then we receive system

$$\dot{x}=Ax + \varphi(x), \tag{1.6.25}$$

where $A = [a_{ij}]$, is a matrix of a linear part, is called a matrix Jacobi or Jakobian, $a_{ij} = \partial F_i/\partial x_j, \quad i,j = \overline{1,n}$; $\varphi(x)$ is a vector of functions $\varphi_i(x)$ which supporting members of decomposition in a row Taylor of function *F(x)* trifles are higher than the first order, meeting a condition

$$\lim_{IIxII \to 0} \frac{II\varphi(x)II}{IIxII} = 0 \,. \qquad (1.6.26)$$

The system of the equations with constant coefficients

$$\dot{x} = Ax, \qquad (1.6.27)$$

is called the system of the first approach for the system of the equations (1.6.25) that is equivalent also for system (1.6.23).

Statements are fair:

1) the trivial solution of system (1.6.25) asymptotically is stability across Lyapunov if all eigenvalues of a matrix *A* (jakobian) of system (1.6.25) have negative material parts, i.e. $Re\lambda_i < 0$ ($i = \overline{1,n}$);

2) if among eigenvalues of a matrix *A* there is at least one root with a positive material part, then the trivial solution of system (1.6.25) is unstability.

In case among eigenvalues there are matrixes *A* there are zero or purely imaginary eigenvalues, then it is impossible to judge stability or instability of the trivial solution of system (1.6.25) on the equations of the first approach. In such cases called critical stability or instability of the trivial decision depends on a nonlinear part φ($x$), and depending on φ($x$) the system (1.6.25) can be stability or unstability.

**B.** *Use of the matrix equation of Lyapunov for a research of stability of systems.*

Very often for the solution of research problems of stability of systems as linear, nonlinear as the convenient tool is used the matrix equation of Lyapunov. At the same time the task can lead as to the algebraic matrix equation of Lyapunov, and the differential matrix equation of Lyapunov.

Stability of the trivial solution of system (1.6.25) or the corresponding linear system (2.6.27) requires also enough that the matrix equation of Lyapunov

$$A^TV+VA=-W, \tag{1.6.28}$$

had positively certain decision $V$, at any positively certain matrix of $W$.

For providing the guaranteed stability degree α > 0, i.e., that $Re\lambda_i(A) \leq -\alpha$ for all $i=\overline{1,n}$ , is necessary also enough that for any set positively certain symmetric matrix of $W$ there would be positively certain matrix of $V$, satisfying to the equation like Lyapunov

$$-2\alpha V+A^TV+VA=-W. \tag{1.6.29}$$

Solutions of the equations (1.6.28) and (1.6.29) are called matrixes of functions of Lyapunov.

We will consider a case when system non-stationary, i.e.

$$\dot{x}=A(t)x, \tag{1.6.30}$$

where $A(t)$ is a matrix which elements depend on time of $a_{ij}=a_{ij}(t), i,j=\overline{1,n}$.

For this case it is also possible to use Lyapunov's equations.

Positive definiteness and negative definiteness of matrixes for a non-stationary case is formulated as follows.

Material continuous function $v(x, t) =x^TV(t)x$ is called positively certain if there is a constant α > 0, such that at all $t$ inequality is carried out

$$v(x,t) \geq \alpha||x|| > 0, \tag{1.6.31}$$

but function $v(x, t)$ is called negatively certain if is α > 0, it that

$$|v(x,t)| \leq -\alpha||x|| < 0, \tag{1.6.32}$$

at all $t$.

The concept of square function of Lyapunov for system (1.6.30) is entered if it is carried out the following conditions:

1) there is positively definitely $v(x, t)$ and α > 0, it that

$$v(x,t) \leq \alpha||x||^2 \ ; \qquad (1.6.33)$$

2) a derivative on time of function *v*(*x, t*), $\dot{v}(x,t)$ owing to the equation (1.6.30) is negatively certain function, i.e. exists α > 0, it that

$$\dot{v}(x,t) \leq -\alpha||x||^2 < 0. \qquad (1.6.34)$$

At the same time existence of square function of Lyapunov is equivalent to exponential stability of system, i.e. statements are fair:

1) if for system (1.6.30) there is a square function of Lyapunov of *v(x, t),* it is system evenly asymptotically it is stability;

2) in order that, the system (1.6.30) asymptotically was stability, enough, that there was a solution of the differential matrix equation of Lyapunov

$$\dot{v}(t) + \mathrm{A}^T(t)V(t) + V(t)A(t) = W(t), \qquad (1.6.35)$$

in the form of positively certain matrix of *V(t)*, at any negatively certain matrix of *W(t)*.

**G.** *Method of vector functions of Lyapunov.*

Further development of the second method of Lyapunov in the theory of stability was introduction to the theory and practice of a research of stability of systems of a so-called method of vector functions of Lyapunov (*VFL* method).

Need of effective use of this method was the problem of a research of stability of the difficult systems of a high order consisting of a set of the subsystems interconnected among themselves. These systems, as a rule, have hierarchical structures.

At a research of such systems, the approach based on decomposition use of the isolated separate subsystems, aggregation and a research of the aggregated system in general is used.

At the same time decomposition – consists in the partition of difficult system of a high order on a number of subsystems, smaller dimension with allocation of influence of interrelations between them. The mathematical model of each subsystem is presented in the form of systems of the equations, in each of which the parts relating to this subsystem and to interrelations are allocated.

Let the difficult system be considered:

$$\dot{x}=F(x,t), \tag{1.6.36}$$

where $x \epsilon R^n$ is a vector of a condition of system; $F(x, t)$ is vector function.

Let now the system (1.6.36) can be presented in the form of set of subsystems, such that are described by the systems of the equations, taking into account their interrelations:

$$\dot{x}^{(i)} = F_i\left(x^{(i)}, t\right) + \sum_{\substack{j=1 \\ j \neq i}} f_{ij}\left(x^{(i)}, x^{(j)}, t\right), \tag{1.6.37}$$

where $i,j=\overline{1,\tau}$; $x^{(i)} \epsilon R^{n_1}$ is a vector of a condition of $i$ that subsystem; $F_i(.)$ is a vector of function of dimension of $n_i$, the characterizing own dynamic properties of $i$ that subsystem, $x^{(i)}$ depending on variable states only this subsystem; $f_{ij}(.)$ is a vector function of interrelations of a subsystem of i with other subsystems of $j \neq i$.

Thus, the difficult system (1.6.36) is represented as association τ is subsystems. Decomposition is carried out so that coordinates (variables) of subsystems weren't crossed, i.e. $n_1 + n_2 + \ldots n_\tau = n$.

There are two approaches to decomposition:

1) the structure of dekompozition difficult system, as much as possible reflects real structure (engineering approach);

2) in case of strong communications of separate parts of difficult system use formalistic approach when decomposition is carried out by regroupings of variables in the equations of system (1.6.36) and reduction to a view (1.6.37) with small $f_{ij}(.)$.

Obviously, decomposition by that will be more effective, than communications of $f_{ij}(.)$ are weaker.

For linear difficult systems

$$\dot{x} = A(t)x, \tag{1.6.38}$$

decomposition on subsystems can be presented in the form

$$\dot{x}^{(i)} = A_i(t)x^{(i)} + \sum_{\substack{j=1 \\ j\neq i}} A_{ij}(t)x^{(j)}. \tag{1.6.39}$$

In this case interrelations of subsystems will be weak if norms of matrixes of $A_{ij}$ (t) are small in comparison with norms of matrixes of $A_i(t)$. Therefore decomposition it is carried out it is iterative, studying elements and blocks of matrixes $A_i(t)$ and $A_{ij}(t)$ (and similar matrixes or jakobian in nonlinear systems), transforming consistently the equations to a look when $f_{ij}$ or $A_{ij}$ will be small or equal to zero.

Sometimes in systems (1.6.36) allocate small parameters at derivatives and carry out decomposition on subsystems with strongly differing time scales (polyspeed parts) further use a so-called method of singular perturbations.

At the following stage after decomposition, at the beginning neglecting interrelations ($f_{ij}(.) \equiv 0$, $A_{ij}(\text{t}) \equiv 0$) we will receive the isolated subsystems

$$\dot{x}^{(i)} = F_i(x^i, t). \tag{1.6.40}$$

The analysis of stability of each isolated subsystem (1.6.40) can be made by the second method of Lyapunov with use of the classical results described above.

For an analysis stage of stability of the isolated subsystems, the aggregation stage and associations of subsystems in difficult system taking into account interrelations follows. The traditional way and joint consideration of family of subsystems this leads consideration of all subsystems with all interrelations to restoration of multidimensionality and difficulties which have been bypassed in the dekompozition analysis of tasks.

In the last decades methods of approximate and estimated aggregation intensively developed. Initial subsystems were replaced with simpler systems, and the aggregated model of some approach to real difficult system turned out. If at the same time it was guaranteed that a number of properties (for example, stability) takes place in initial system when these properties are available in the simplified system, or the divergence between them is in the admissible set limits, then aggregation is called estimated. In the theory of stability of difficult systems it is required that from stability of the simplified system stability (in a sense) of initial (real) system and that processes in the simplified system majority a certain sense processes in initial (real) difficult system followed. The simplified systems meeting such requirements are called *systems (or models) comparisons*.

One of the most developed methods of creation of scalar systems of comparison is based on use of functions of Lyapunov and a method of comparison when the initial system (2.6.36) is majorized by some system

$$\dot{y} = \notin(y, t), \tag{1.6.41}$$

such that positively certain function *υ(x, t)* which derivative owing to system (1.6.36) satisfies to differential inequality $\dot{\upsilon}(x,t) \leq \notin(\upsilon(x,t),t)$, in some area

$$(x,t) \in G, \tag{1.6.42}$$

where $\notin(\cdot)$ is material, continuous function, such that through each point $(t_{0,}y_0)$, passes the only decision $y(y_{,}t_{0,}t)$ the equations (comparison) (1.6.42).

It is in that case proved that all properties of stability (1.6.36) are defined by the corresponding stability of system (1.6.42).

This method is applied to the isolated subsystems, then the set of functions of Lyapunov for subsystems is used for assessment of stability of all difficult system taking into account interrelations. One of the main ideas which have led to introduction of a concept of vector function of Lyapunov as sets of "subsystem" functions of Lyapunov consists in it. It was for the first time offered the famous mathematician Richard Bellman in work of 1962. Almost at the same time similar idea was is published in V.M. Matrosov's work in which it is offered to unite a method of functions of Lyapunov and inequality like Chaplygin, for receiving vector function of Lyapunov and the vector system of comparison.

Further the term the vector function of Lyapunov (*VFL*) was widely included into literature according to the theory of stability of difficult systems, and the *VFL* method became the main method of estimated aggregation in difficult systems.

On the basis of decomposition, the analysis of subsystems and estimated aggregation by means of *VFL* very effective method to a research of stability of difficult dynamic systems of a high order of the different physical nature is received.

**D.** *The theory of stability based on production of entropy.*

The elements of the theory of stability stated in the previous sections of this chapter belong to the general bases of the theory of stability, without concerning concrete systems. But the concrete fields of science have the certain specifics which have to be considered by considerations of questions of stability. Methods of researches of stability in the concrete fields of science carried the history and considered specifics of objects of these sciences.

So in thermodynamics there are specific theories stability which we will review in this section briefly.

The classical theory of stability of thermodynamic systems is the theory of stability of Gibbs-Dyugem which is defined by conditions:

$$d_F = -Td_iS \leq 0,\ (T, V, N_R = const); \quad (1.6.43a)$$

$$d_G = -Td_i S \leq 0,\ (T, p, N_R = const); \quad (1.6.43b)$$

$$d_H = -Td_i S \leq 0,\ (S, p, N_R = const); \quad (1.6.43c)$$

where $F, G, H$ are respectively potentials of free energy of Helmholtz, free energy of Gibbs and an enthalpy; $T$, $V$, $N_R$, $p$, $S$ are respectively temperature, volume, number of moths, pressure and entropy. At the same time the enthalpy of $H$ is the function of a state determined by variables of a condition of system

$$H=U+pV, \tag{1.6.44}$$

where $U$ is a variable of energy of system.

In equilibrium state the thermodynamic system has to remain steady concerning any fluctuations and perturbations of the external environment.

The classical theory of Gibbs-Dyugem considers stability of an equilibrium condition of thermodynamic system. In this theory it is claimed that equilibrium state is steady against any perturbation if it leads to reduction of entropy of system. Equilibrium state it is steady against fluctuations if also the entropy at the same time decreases, i.e. fluctuation is extinguished.

We will consider for the isolated system types of stability of thermodynamic system, from a position of the classical theory of Gibbs – Dyugem.

*Thermal stability.*

Fluctuation of temperature of $T$ in some isolated system is considered.

Conditions of thermal stability of equilibrium state has an appearance

$$\frac{1}{2}\delta^2 S = -\frac{C_V(\delta T)^2}{2T^2} < 0, \tag{1.6.45}$$

where $\delta^2 S$ is the second component in decomposition of entropy in power series of Taylor

$$S = S_0 + \delta_S + 1\delta^2 S + \cdots; \tag{1.6.46}$$

$C_V$ is molar thermal capacity in the considered constant volume of the environment; $\delta T$ is small change of temperature in volume of (Fig. 1.10).

From (2.6.45) $C_V > 0$ is required that is carried out as thermal capacity at the constant volume is positive.

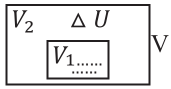



Fig. 1.10.

ΔU is perturbation of energy from one part $V_1$ to another $V_2$ , causing changes of temperature at a size $\delta T$.

*Mechanical stability.*

We will consider resistance of thermodynamic system to fluctuations of volume of a subsystem at the remaining invariable number of moths of *N*. The system divided into two parts (fig. 1.10) is considered, but here it is supposed that in parts $V_1$ and $V_2$ occur small changes of volumes $\delta_{V_1}$ and $\delta_{V_2}$. The full volume of system doesn't change therefore $\delta_{V_1} = \delta_{V_2} = \delta_V$.

In this case, stability conditions takes a form

$$\delta^2 S = -\frac{1}{TR_T} \cdot \frac{(\delta_V)^2}{V} < 0, \qquad (1.6.47)$$

where $V$ is any volume is $V_1, V = V_1; R_T$ is coefficient isothermal compressibility $R_T = -(1/V)^{dV}/_{dp}$ . $T$ is considered constant $T$=const.

The ratio (1.6.47) is fair at $R_T > 0$ that is carried out.

In case of $R_T < 0$, the system is in an unstable nonequilibrium state.

*Resistance to fluctuations of number of mols.*

Fluctuations of number of mols of various components of system are caused by chemical reactions and the phenomena of diffusion.

*Chemical stability.*

Fluctuations at the same time are defined as fluctuations of degree of completeness of reaction ξ concerning their equilibrium values. At the same time *T*=const is also supposed.

In that case, the condition of stability takes a form

$$\frac{1}{2}\delta^2 S = \frac{1}{2T}\left(\frac{dA}{d\xi}\right).\ (\delta\xi)^2 < 0, \tag{1.6.48}$$

where *A*, so-called affinity:

$$A/_T=(dS/_{d\xi})_{U,V}\ , \tag{1.6.49}$$

(dA/dξ) is value in an equilibrium state.

At course of several reactions of a condition (1.6.48) becomes complicated:

$$\frac{1}{2}\delta^2 S = \sum_{i,j}\frac{1}{2T}\left(\frac{dA_i}{d\xi_j}\right)_0 S\xi_i S\xi_j < 0. \tag{1.6.50}$$

*Resistance to fluctuations, caused by diffusion.*

Fluctuations are also possible because of exchange of substance between parts of system.

Stability conditions of an equilibrium state it is presented in the following form

$$\delta^2 S = -\sum_{i,j}\left(\frac{d}{dN_j}\frac{M_{1j}}{T}\right)SN_i\ SN_j < 0, \tag{1.6.51}$$

where $M_{1j} = -T\cdot(dS/_{dN_1})$ is chemical potential in volume of $V_1$, $SN_i$ , $SN_j$ are changes of number of mols in volumes of $V_1$ and $V_2$.

Thus, the general condition of resistance of an equilibrium state to thermal, volume fluctuations and also to fluctuations of number of the mols caused chemical reactions and diffusion is expressed by the following ratio:

$$\delta^2 S = -\frac{C_V(\Delta T)^2}{T^2} - \frac{1}{TR_T}\cdot\frac{(\delta V)^2}{V} - \sum_{i,j}\left(\frac{2}{2N_j}\cdot\frac{M_j}{T}\right)SN_i SN_j < 0, \tag{1.6.52}$$

where system $C_V$ is thermal capacity with any capacity of $V$ and chemical potential of $M_j$.

The considered theory of stability of Gibbs-Dyugem is fair only under certain conditions, for example $T$=const. From this shortcoming the general approach to stability of thermodynamic systems based on production of entropy which can be used also for the analysis of stability of nonequilibrium systems is free.

*Stability of thermodynamic systems on the basis of production of entropy.*

So, the task to define stability on the basis of receiving expression for production of entropy, caused by fluctuations is set. Obviously, the system will be stable against fluctuations if it the corresponding production of entropy is negative, i.e.

$$\Delta_i S < 0. \qquad (1.6.53)$$

The general expression of production of entropy takes a form

$$\frac{d_i S}{dt} = \sum_{k} F_R \frac{dx_R}{dt} = \sum_{k} F_R J_R \quad , \qquad (1.6.54)$$

where $F_R$ is thermodynamic force; $\frac{dx_R}{dt} = \mathcal{J}_{R,}$ - a thermodynamic stream.

Thermodynamic forces arise because of not uniformity of temperature, pressure or chemical potential. If to designate through $I$ and through $F$ according to a state equilibrium and a state because of fluctuation, then we have

$$\Delta_i S = \int_I^F d_i S = \int_I^F \sum_{k} F_{k} \, dx_{k}. \qquad (1.6.55)$$

We will consider stability of an equilibrium state in the beginning.

Then, we will have:

1) for chemical stability:

$$\Delta_i S = \left(\frac{\partial A}{\partial \xi}\right)_0 \frac{(S\xi)^2}{2T} < 0 \, , \qquad (1.6.56)$$

or in a case $k$-chemical reactions

$$\Delta_i S = \sum_{i,j}^{k} \frac{1}{2T} (\frac{\partial A_i}{\partial \xi_j})_o^\delta \xi_i \xi_j < 0 \; ; \qquad (1.6.57)$$

2) for thermal stability:

$$\Delta_i S = -\frac{C_V}{T_0^2} \cdot \frac{(\delta T)^2}{2} < 0 \; ; \qquad (1.6.58)$$

3) generally:

$$\Delta_i S = \int_I^F \sum_k F_k dx_k = -\frac{C_V (ST)^2}{T^2} - \frac{1}{T k_T} \frac{(\delta V)^2}{2V} - \sum_{i,j} \left( \frac{\partial}{\partial N_j} \frac{\mu_i}{T} \right) \frac{\partial N_i S N_j}{2} < 0 \, , \qquad (1.6.59)$$

or in designations $\delta^2 S$ we have:

$$\delta^2 \mathrm{S} < 0, \frac{1}{2} \frac{d\delta^2 \mathrm{S}}{\mathrm{dt}} = \sum_k \delta \mathrm{F}_k \delta \mathrm{J}_k > 0. \qquad (1.6.60)$$

These equations follow as a special case from the general theory of stability of Lyapunov.

We will consider stability of *nonequilibrium* stationary states now.

Stationary states in the linear mode are states with extreme values of production of entropy.

Near balance in the linear mode in steady systems ratios are carried out:

$$\rho = d_i S / dt > 0, \qquad (1.6.61)$$

$$\frac{d\rho}{dt} = \frac{2}{T^2} \sum_{ij} (\frac{\partial A_i}{\partial \xi_j}) \vartheta_i \vartheta_j < 0, \qquad (1.6.62)$$

where $\vartheta_i, \vartheta_j$ are variable:

$$\vartheta_k = \sum_i \mathcal{L}_{ki} (A_i / T) \, , \qquad (1.6.63)$$

$\mathcal{L}_{ki}$ is coefficient of the so-called phenomenological equations connecting reaction speeds $\vartheta_k$ and $A_i/T$.

$$\dot{\vartheta}_1 = \mathcal{L}_{11} \frac{A_1}{T} + \mathcal{L}_{12} \frac{A_2}{T}, \qquad (1.6.64a)$$

$$\dot{\vartheta}_2 = \mathcal{L}_{21}\frac{A_1}{T} + \mathcal{L}_{22}\frac{A_2}{T}. \quad (1.6.64b)$$

Conditions (1.6.61) and (1.6.62) guarantee stability of nonequilibrium stability state in the linear mode near balance and make stability conditions across Lyapunov.

Stability of the systems *far from balance.*

The systems subject to a stream of energy and substance can pass into the states far from thermodynamic balance, into "the nonlinear mode".

In the nonlinear mode thermodynamic flows of $I_d$ aren't linear functions of thermodynamic forces of $F_d$ . In systems nonequilibrium (nonlinear) as a result of fluctuations or other small perturbations there is a transition from an unstability state to one of possible new states. These new states can be high-organized.

For *non equilibrium* stationary states the most general way of a research of stability is use of the second method of Lyapunov which is considered by us in the previous sections of this chapter.

At the same time the equations of system can be written down in a usual look, or in private derivatives if $x_i$ depend not only on time, but also on spatial coordinates, or other variables related. And in this case stability conditions of thermodynamic system are also defined by conditions:

$$\mathcal{L}(x) > 0, \dot{\mathcal{L}}(x) < 0\,, \quad (1.6.65)$$

where $L(x)$ is Lyapunov's function. But in case the $x_i$ *(t)* variables are functions of coordinates (for example, in nonequilibrium systems it is concentration of the $n_k$ components), $L(x)$ is called Lyapunov's *functionality*.

The condition of stability of nonequilibrium steady state with use of functionality of Lyapunov of $L=-\delta^2S$ has an appearance

$$\frac{d}{dt}\frac{\delta^2 S}{2} = \sum_{k} \delta F_{k}\, \delta \mathcal{J}_{k} > 0. \quad (1.6.66)$$

This condition is a sufficient, but necessary condition of stability, i.e. if (1.6.65) isn't carried out, that the system can be unstability, at the same time

$$\sum_{k} \delta F_{k} \, \delta \mathcal{J}_{k} < 0,$$

there is a necessary, but not sufficient condition of instability of system.

For thermodynamic systems the method of linear approach by Lyapunov's method can be also used.

## 1.7. Fractals

**A**. *Fractals.*

*Fractals* is new the concept entered into science by Benoit Mandelbrot in the late sixties of the 20th century. Fractals are the complex geometrical structures having "self-similarity" and described by nonintegral dimension.

B. Mandelbrot called *a fractal* a set for which his Hausdorff's dimension more topological is strict:

$$d_H > d_T \; . \tag{1.7.1}$$

We will define a Hausdorff's and topological dimension.

The *topological dimension* $d_T$ is it dimension of geometrical objects in the usual sense when, to a calculating set (a point or points) attribute dimension zero, to lines to straight lines and curves it is dimension one $d_T = 1,$ to surfaces have dimension to $d_T = 2$, volumes have dimension of $d_T = 3$, etc. Intuitively it not always arranges to eat, for example, curves single in some surface, and there are curves which are almost covering a surface and these curves in a usual metrics have identical dimensions of $d_T = 1$ (Fig. 1.11)

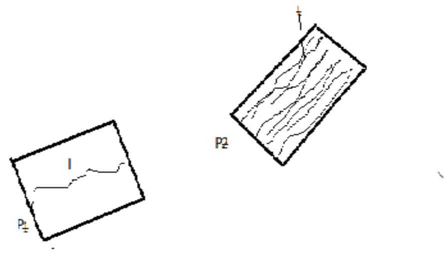



Fig. 1.11.

For assessment of degree of complexity of geometrical objects or for the characteristic of degree of complexity of trajectories of particles in phase space φ. Hausdorff has entered a new measure or Hausdorff's dimension of $d_H$ (sometimes call dimension Bezikovich – Hausdorff ) as follows.

Some set which points are shipped in spaces of some dimension of $d_T$ is considered. This set becomes covered by $n$ measured cubes, densely packing them. Kubes undertakes so much how many all considered set (Fig. 1.12) is necessary for a covering them.

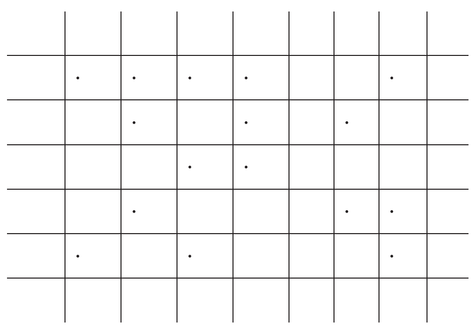


Fig. 1.12.

We will designate the party of a cube through $r$ and number of cubes to which one point of a set through $N(r)$ gets at least.

Then the Hausdorff's dimension of the considered set is equal

$$d_H = \lim_{r-0} \frac{\ell n\, N(r)}{\ell n\left(\frac{1}{r}\right)}. \qquad (1.7.2)$$

Is easy to calculate d that for a piece of direct or smooth curve $d_H = d_T = 1$, and for a part of the $d_H = d_T = 2$ plane, etc. i.e. for habitual, everyday occurences of a Hausdorff's and topological dimensions coincide.

We will consider other cases when $d_H \neq d_T, d_H > d_T$.

Classical example is the so-called curve of Koch.

The curve turns out as follows. The piece of single length undertakes is divided into three and is thrown out from of that 1/3 a part in the middle. Together an average piece two parties (length on 1/3 every) an equilateral triangle are under construction.

Thus, there will be fourth links 1⁄3 long everyone, so, that the total length of a broken line will be 4⁄3.

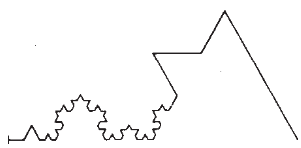


Fig. 1.13.

With each of four pieces of the formed broken line we arrive in the same way, i.e. we throw out the third part in the middle and we build on a broken line from two links. We continue this operation more and more time, etc. After many iterations the broken line will become very twisting (Fig. 1.13). In a limit at infinitely bigger number of steps we receive a continuous nowhere differentiable curve.

On $n$ number a step length of a piece of a broken line is equal

$$r_n = (^1/_3)^n. \qquad (1.7.3)$$

These pieces also play a role of the "cubes" covering the formed Koch's curve. Also the number of such "cubes" is also just counted:

$$N(r_n) = 4^n, \quad (1.7.4)$$

from here, hausdorff's dimension of a curve of Koch:

$$d_H = \ell_n 4/\ell_n 3 \approx 1.26. \quad (1.7.5)$$

So $d_H > d_T$.

The second classical example, is an example of a Cantor set. This set is called in honor of the great German mathematician George Kantor opening him in 1883. This set plays a large role in modern nonlinear dynamics.

The Cantor set is under construction as follows (Fig. 1.14):


Fig. 1.14.

Average jumps out of a single piece 1⁄3 part. Also we treat each of the formed two pieces, etc. step by step. What, remains from a piece after infinite number of steps and makes a Cantor set. Length of the pieces which are thrown out 5h is equal

$$l = 1/3 + 2/9 + 4/27 + \cdots = 1. \quad (1.7.6)$$

Thus, total "length" of the remained Cantor set is equal to zero and therefore, for him $d_T = 0$. However the Hausdorff's dimension of a Cantor set will be equal

$$d_H = \frac{\ell n2}{\ell n3} \approx 0{,}63092\ \ldots, \quad \text{т.е.} d_H > d_T. \quad (1.7.7)$$

For a classical example of Brownian motion on the plane $d_H = 2 > d_T = 1$.

So, Mandelbrot has determined by a fractal a set which has a Hausdorff's dimension more topological (1.7.1), $d_H > d_T$.

Thus all reviewed classical examples are examples of fractals. Examples with Brownian motion also the condition (1.7.1) is satisfied though dimension $d_H$ is whole. Therefore definition of a fractal as sets of fractional dimension strictly speaking it isn't always right, i.e. it is possible to enter specification in (1.7.1) in a look, fractals it is such sets which satisfy to a ratio:

$$d_T > d_H \leq d_T + 1, \qquad (1.7.8)$$

where $d_T$ is topological dimension of a geometrical set.

Fractals share on regular (like Koch's curve, a Cantor set and. etc.) and stochastic (like a trajectory of Brownian motion).

The main merit in development of fractal geometry belongs to Benoit Mandelbrot. Thus, entered a concept about fractals and fractal geometry into science to B. Mandelbrot. Thanking first of all to his works in many fields of science a concept of fractals have received broad attention and application in the description of various phenomena and structures showing chaotic properties.

**B.** *Fractal dimensions.*

There are various types of fractal dimensions. The given type of dimension according to Hausdorff is higher $d_H$ is usually is called *capacitor* dimension.

Other examples of sets for which it is possible to calculate capacitor dimension of $d_H$ , except above-stated it the sets arising at the “horseshoe” map and transformation of "baker".

Horseshoe map and transformation of the baker us have been considered in the report of last year, and represent the simplest examples of iterative dynamic processes on the plane which lead to loss of information and fractal properties.

Calculation of capacitor fractal dimension is made for horseshoe map how it is calculated in the previous examples of a curve of Koch and others.

According to (1.7.2) we receive

$$d_H = \frac{\ell n 2}{\ell n|\mathrm{r}|} + 1, \qquad (1.7.9)$$

where $r$ is compression parameter, $0 < r < {}^1/_2$ .

For transformation of the "baker" capacitor dimension

$$d_H = \frac{\ell n 2}{\ell n|\lambda|} + 1, \qquad (1.7.10)$$

where $\lambda$ is transformation parameter, $|\lambda|>2$.

Though determination of capacitor fractal dimension has simple

interpretation as the measure is a geometrical measure of a covering cubes or spheres of a usual geometrical object, but she has certain shortcomings: first, connected with geometricity, i.e. she doesn't consider the frequency with which the trajectory visits a covering element (at chaotic structures), secondly, calculation of the hyper cubes forming set coverings in phase space demands very big expenses of computing time. But in realities, determination of fractal dimension with the help, for example numerical methods in fact it is never made on infinite (very large number of iterations) a set, and the number of the covered points is limited to some size $N_0$ . Therefore for final number of points there is always the minimum number of $r_{\min}$ behind which at reduction the quantity $N(r_n)$ ceases to change, reaching some $N_0$ value.

Therefore, except capacitor dimension of $d_{\mathrm{H}}$ there are also alternative types of fractal dimensions which yield numerical results rather close each other and to capacitor dimension.

The *pointsion dimension* is entered as follows.

The trajectory in phase space throughout a long interval of time is considered (Fig. 1.15.)

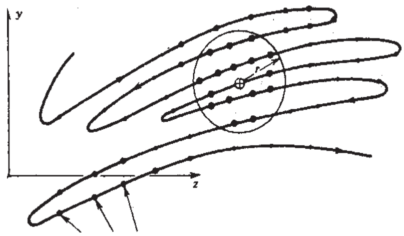



Fig. 1.15.

Selection of points is made to receive rather large number of the representing points on a trajectory, further around some point on a trajectory the sphere of radius of *r* is described (or a cube with *r* edge) and the number of selective points of *N(r)* which have got in a cube is counted (or spheres). The probability that the selective point will appear in the sphere, is determined by a formula:

$$\rho(r) = \frac{\mathrm{N(r)}}{\mathrm{N_o}}, \qquad (1.7.11)$$

where $N_0$ is total number of selective points *d*.

The pointsion dimension designated by $d_\rho$ is found to the following ratios:

$$d_\rho = \lim_{r\to 0} \frac{\ell_n \rho(r,x_i)}{\ell_n r}, \qquad (1.7.12)$$

where $x_i = x(t_i)$ is value of a vector *x* phase coordinates in timepoint of $t_i$ (or *i* in a discrete case); $\rho(r, x_i)$ is the probability, the fact that the trajectory in time interval $\Delta t_i \to 0$ will be is in the sphere of radius of $r \to 0$.

For some attractors this definition doesn't depend on $x_i$ point. But for many attractors of $d_\rho$ depends from $x_i$, therefore use average pointsion dimension.

To receive average pointsion dimension, choose in a random way a set of points of $M<N_0$ and in each his point calculate $d_\rho(x_i)$, further calculated

$$d_\rho = \frac{1}{\mathrm{M}} \sum_{i=1}^{\mathrm{M}} d_\rho(x_i). \qquad (1.7.13)$$

Sometimes average probabilities $P(r,x_i)$. For this purpose the casual subset gets out of *M* points located around an attractor ($M \ll N_o$). Then assuming,

$$\lim_{r \to 0} \frac{1}{M} \sum_{i=1}^{M} \rho(r, x_i) = ar^{d_\rho} ,$$

have

$$d_{\rho=} \ell im \frac{\ell_n(^1/_M)\rho \Sigma P(r)}{\ell n\, r}. \qquad (1.7.14)$$

usually, $N_o \approx 10^3 \div 10^4$, $M \approx 10^2 \div 10^3$.

Different way calculation pointsion fractal dimension averaging on radiuses spheres (or size cubes) phase space containing same quantity (for example some $N$ points). Choosing various reference $x_i$ (the centers or cubes), calculates $r_i(N)$ take to points:

$$\bar{r}(N) = \frac{1}{n} \sum_{i=1}^{n} r_i\,(N). \qquad (1.7.15)$$

*Correlation dimensions* defined as follows. Well when determining continuous basic trajectory sampled, i.e. replaced with a set from $N$ points $\{x_i\}$ space. Distance between couples $S_{ij} = |x_i x_j|$, using either usual euclidean measure distance, other form norm vector ||.||.

Correlation function defined

$$R(r) = \lim_{N - \infty} \frac{1}{N^2} \big(M(i\,, j){:}\; S_{ij} < r\big), \qquad (1.7.16)$$

where $M(i,j$ is number pair points $(x_i x_j)$, for which $S_{ij} = ||x_i - x_j|| \le r$.

For many attractors

$$\lim_{N - \infty} R(r) = ar^d,$$

owing determined by formula:

$$D_r = \lim_{r - 0} \frac{\ell n R(r)}{\ell n r}. \qquad (1.7.17)$$

But $R(r)$ can be more effectively at sphere (or cube) described around each point $x_i$, i.e. calculated

$$R(r) = \lim_{N\to\infty} \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} M\left(r - \left\|x_i - x_j\right\|\right), \qquad (1.7.18)$$

where $M(\cdot)=1$ at $(\cdot)>0$, $M(\cdot)=0$, at $(\cdot)<0$.

*Information fractal dimension* proceeds similarly capacitor (hausdorff's) fractal dimension, but taking into account the frequency with which the trajectory gets to a covering element is the sphere (or a cube). Here too the set of points which fractal dimension needs to be defined becomes covered by $N$ spheres or cubes with a radius or an edge of $r$. In turn the set of points is considered as uniform sampling of a continuous trajectory.

For calculation of information fractal dimension there is a number of points of $N_i$ in which of $N$ cells of a covering and we estimate probability of $P_i$ to find $i$ point in a cell:

$$P_i = \frac{N_i}{N_o}, \quad \sum_{i=1}^{N} P_i = 1, \qquad (1.7.19)$$

where $N_0$ is the total number of points in a set.

It is known that information entropy is on expression:

$$I_r = -\sum_{i=1}^{N} P_i \, \ell n \, P_i. \qquad (1.7.20)$$

at small $r$: $I_r \approx d_i \ell n(1/r)$,

therefore information fractal dimension is determined by a formula:

$$d_I = \lim_{r\to 0} \frac{I(r)}{\ell n\left(\frac{1}{r}\right)} = \lim_{r\to 0} \frac{\Sigma P_i \ell n \, P_i.}{\ell n \, \varepsilon}. \qquad (1.7.21)$$

Generally the ratio between information and capacitor dimensions is satisfied to inequality:

$$d_I \leq d_H. \qquad (1.7.22)$$

There is one more type of fractal dimension which is entered on the basis of Lyapunov's indicators and is designated by $d_\lambda$ therefore it is possible to call *Lyapunov's fractal dimension*.

It is known that Lyapunov's indicators characterize the speed of their running from each other, and for trajectories out of an attractor is the speed of their approach to an attractor.

If $\lambda_{\mathcal{L}i}$ are Lyapunov's indicators, then when streamlining

$$\lambda_{\mathcal{L}1} > \lambda_{\mathcal{L}2} > \cdots > \lambda_{\mathcal{L}n}, \quad (1.7.23)$$

and $\lambda_{\mathcal{L}k}$ is such indicator that

$$\sum_{i=1}^{k} \lambda_{\lambda i} \geq 0,$$

that the Lyapunov's fractal dimension is entered on a formula:

$$d_{\lambda\mathcal{L}} = k \overline{\bigoplus} \frac{\sum_{i=1}^{k} \lambda_{\mathcal{L}i}}{\lambda_{\mathcal{L}k}}. \quad (1.7.24)$$

The following states between various fractal dimensions are fair:

$$d_r \leq d_i \leq d_H, \quad (1.7.25a)$$

$$d_{\lambda\mathcal{L}} \leq d_H. \quad (1.7.25b)$$

In many cases for standard (known) strange attractors all values of different types of fractal dimensions are very close.

For example, in case of transformation of the baker, it is established that

$$d_I = d_{\lambda\mathcal{L}} \leq d_H = 1 + \frac{\ell n 2}{\ell n |\lambda|}. \quad (1.7.26)$$

In the conclusion of this section we will note that all fractal dimensions of standard strange (chaotic) attractors nonintegral number, moreover irrational number.

**B.** *Fractals and chaos.*

Chaos or the chaotic movements (strange attractors) in essence have fractal structure as trajectories in chaotic structure of a stokhasticly.

The fractal geometry in nonlinear dynamics is applied in two purposes:

1. for determination of strangeness of attractors (randomness);
2. for measurement of fractal dimension.

At numerical calculations and physical experiments the fractal dimension and Lyapunov's indicators are defined, sampling an object and signals the sequence equidistant (on time) points and processing the obtained data on the computer.

Three main methods are known:

1) temporary samplings of variables in phase space;

2) calculation of fractal dimension of maps of Poincare;

3) creation of pseudo-phase space on measurements of one variable.

Variables are measured in the first and third methods through identical periods { $x(t_0+Tn)$ } where $n$ is integers, a time interval of $T$ is chosen so that he made a certain share of the period of the compelled force. In case of the second method and if Poincare's map is carried out on time, then $T$ is the period of trajectories. If Poincare's map is carried out on any other variables in phase space, then data correspond to various time points, depending on the chosen Poincare's map.

In most cases, at calculation of fractal dimension, are used from several thousand to tens of thousands of points. Direct algorithms for calculation of fractal dimension on $N_0$ to points contain $N_o^2$ of operations and demand for calculations of supercomputers. When using special programs lower only operations to $N\ lnN_0$.

Values of frontal dimensions of different types of standard strange attractors are given in the following table.

Table 1.1

| NoNo | Name of system | Type of dimension | Dimension size |
|---|---|---|---|
| 1. | Lorentz's system | Capacitor $d_{\text{H}}$ | 2.06±0.01 |

| | | | |
|---|---|---|---|
| | | Correlation $d_R$ | 2.05±0.01 |
| | | Lyapunov’s $d_{\lambda L}$ | 2.07 |
| 2. | Ryossler's system | Lyapunov’s $d_{\lambda L}$ | 2.01 |
| 3. | Henon’s map (a = 1,4; b = 0.3) | Capacitor $d_H$ | 1.26 |
| | | Correlation $d_R$ | 1.21±0.01 |
| | | Lyapunov's $d_{\lambda L}$ | 1.26 |
| 4. | Logistic map | Capacitor $d_H$ | 0.538 |
| | | Correlation $d_R$ | 0.500± 0.005 |
| 5. | Circuit (system) of Chua | Lyapunov's $d_{\lambda L}$ | 2.82 |

## CHAPTER 2. THEORY OF DYNAMIC SYSTEMS

Bases of the modern theory of dynamic systems have been developed by the great French mathematician Henri Poincare. The theory of dynamic systems researches types of dynamic behavior of the systems described by the difficult nonlinear equations. The theory of dynamic systems is fundamental mathematical discipline, closely connected with many fields of mathematics. Concepts, methods and submissions of the theory of dynamic systems strongly stimulate researches in many other branches of science, moreover leads to emergence of the new directions of sciences, so for example, applied dynamics, nonlinear dynamics or the theory of Chaos. The theory of dynamic systems includes a number of the main disciplines in particular, finite-dimensional differential dynamics. The last is closeness connected with such disciplines as the ergodic theory, symbolical dynamics and topological dynamics. The modern theory of dynamic systems rather extensive in this section we will consider some provisions of the theory.

**Basic concepts.** The theory of dynamic systems, first of all, includes the following elements:

1) phase space of $X$ which elements "points", represent possible conditions of system;

2) "time" which can be both continuous, and discrete. Time can change only in one direction, in the future (irreversible processes), or in two directions both in the past and in the future (reversible processes);

3) law of evolution of system.

In the general formulation this such edited (description) which allows to define a condition of system in each time point of $t$, knowing a state at all previous moments. Thus, the most general law of evolution of system can depend on time of $t$ and has infinite memory.

So, if the system was in some state $x$, during $t$ it will pass into a new state which unambiguously is defined by values $x$ and $t$, i.e. the new state is function of two variables $F(x, t)$. Fixing $t$, we receive transformation $\varphi_t$: $x \rightarrow F(x,t)$ phase space in. At the same time transformations $\varphi_t$ form semi-group, i.e.

$$\varphi(t_1+t_2) = \varphi(t_1)+\varphi(t_2). \qquad (2.1.1)$$

For reversible system, transformations $\varphi^t$ are defined both for positive, and for negative values $t$, and each transformation $\varphi^t$ is reversible. Thus, the reversible dynamic system with discrete time is represented cyclic group $\{ F = (\varphi^1) / n \in Z \}$, biunique transformations of phase space to itself, and the reversible dynamic system with continuous time determines one-parametrical group $\{ \varphi^1 / t \in R \}$ biunique transformations $x$ to itself.

In the theory of dynamic systems in the center of attention there is a problem of studying of asymptotic behavior, i.e. behavior of system at aspiration of time of infinity.

Historically interest in smooth dynamic systems with continuous times has been attracted from opening of that fact by Newton that the movement of mechanical objects can be described by the ordinary differential equations of the second order. But also many objects of other sciences it is described by the ordinary differential equations of various orders. Therefore historically dynamic systems, were identified with the systems described by the ordinary equations in the beginning.

Almost phase space of dynamic system has a certain structure of a constant in time.

Now there are various theories studying the dynamic systems keeping the structures.

These are the following theories:

1) The *ergodic theory* in which the phase space $X$ is space with a measure, i.e. Lebesque's space with final or so-called her a final measure of $M$.

The ergodic theory goes back the roots to an ergodic hypothesis of Boltzmann which for the systems of statistical mechanics postulates equality of some temporary and spatial averages. Systematic development of the ergodic theory as section of mathematics has been begun in the 30th years of 20th century by J. fon Nejman. It is continued by J.D. Birkgof, E. Hopf, etc. Modern development has been more connected with works of the outstanding Soviet mathematician A. Kolmogorov and his pupils Ya. Sinay, V. Rokhlin and others;

2) *topological dynamics* in which the phase space is the topological space metrized compact or locally compact. Topological dynamics considers groups, homeomorphisms and semi-group of continuous transformations of such spaces. Sometimes these objects are called topological dynamic systems.

The foundation of topological dynamics has been laid by Henri Poincare, at the qualitative solution of the differential equations which can't be solved analytically. The big contribution to the theory of topological dynamics was made by M. Morse and J.D. Birkgof.;

3) the theory of smooth dynamic systems, or *differential dynamics* when the phase space has structure of smooth variety, for example, is area or the closed surface in Euclidean space.

This theory studies diffeomorfizm and streams (smooth one-parametrical groups of diffeomorfizm) on such varieties and iteration of irreversible differentiable mappings.

At a finite-dimensional case the smooth variety possesses natural locally compact topology, the theory of smooth dynamic systems naturally uses concepts and results of topological dynamics. Other reason of dependence of differential dynamics on topological consists that when studying asymptotic behavior of smooth dynamic systems often there are very difficult rough phenomena. For example, some important invariant sets of smooth systems, for example, attractors can not have any smooth structure and, therefore, such sets have to be investigated from other, rough point of view.

*Symbolical dynamics*, the area studying a special class of topological dynamic systems which arise as the closed invariant subsets of transformation of shift in space of the sequences.

*Differential dynamics* is also closely connected with the ergodic theory as invariant measures represent the powerful tool for the analysis of asymptotic properties of smooth dynamic systems.

The foundation of differential dynamics has also been laid by H. Poincare. He has emphasized high-quality approach as opposed to traditional attempts to receive obvious solutions of the differential equations of mechanics and also he has created the local theory of reflections and vector fields in the neighborhood of motionless and periodic orbits. Further, at early stages of development of differential dynamics, the big contribution to the theory was made by A.M. Lyapunov and G. Hadamar who have entered various concepts of stability and have developed the fixed analytical assets of researches of stability. Further the contribution was made in development of the theory of differential dynamics by J.D. Birkgof, then A. Denjoy, E. Hopf and others.

Other instrument of modern approach to the analysis of smooth dynamic systems was the concept of structural stability or roughness which at first it is entered and was developed by A.A. Andronov and L.S. Pontryagin for the analysis of streams on surfaces and, further has been developed in M.Peixoto's works with less severe conditions.

The big contribution to the theory was made by S. Smale who has proved that strange attractors like "horseshoe" have structural stability. Further S.Smale, D.Anosov, Ya.Sinai, Dj.Bowen have developed bases of the theory of hyperbolic dynamic systems. The big contribution to the smooth ergodic theory was made by Ya. Sinay and D. Ruelle and others;

*3)Hamilton's*, or *symplectic dynamics*, is natural synthesis of the analysis of the differential equations of classical mechanics. The phase space at the same time represents four-dimensional smooth variety with the closed nondegenerate differential form Ω. One-parametrical groups of the diffeomorfizm keeping a form Ω correspond to the differential equations of classical mechanics in Hamilton's form.

Hamilton dynamics became subject of the analysis of the theory of dynamic systems because of the problems arising in heavenly mechanics. And here the big contribution was made by H. Poincare, having applied essentially new high-quality approach to the analysis of problems Ω bodies. Further Hamilton dynamics was divided into two directions:

1) researches of the dynamic complexity arising because of a certain giperbolicness;

2) the analysis of the integrated systems and their indignations which has led to *KAM* the theory.

The big contribution to development of Hamilton dynamics was made by A. Kolmogorov who has proved that many qualitative features of the integrated systems to some extent remain under the influence of perturbations and also arise in typical situations.

We will stop on definitions of some special terms of the theory of dynamic systems.

1. *Homeomorphism*. The homeomorphism is understood as univocity between two topological spaces at which both mutually return mappings determined by this compliance are continuous. These *mappings* are called *gomeomorfly*, or topological mappings and also homeomorphisms, and the spaces belonging to one topological type are called gomeomorfly, or *topological equivalent*.

2. *Homeomorphisms group*. It is group of gomeomorfly maps of topological space of *X* on itself.

3. *Diffeomorfizm*. It is differentiable homeomorphism, smooth homeomorphism, biunique and continuously differentiable mapping.

4. *Isomorphism.* This compliance (relation) between objects or the systems of objects expressing somewhat *identity* of a structure.

Isomorphism, or isomorphic mapping, systems $A$ on the $A^1$ system biunique map φ sets $A$ on the set of $A^1$ having the following properties is called:

$$\varphi(F_i(a_{1,} \dots a_i))=F_i(\varphi(a_1),\dots,(\varphi(a_{n_i})), \qquad (2.1.2)$$

$$P_j(a_{1,}\dots,am_j) \Leftrightarrow P_j(\varphi(a_1),\dots,(\varphi(a_{n_i})),$$

for all elements $a_1, a_{2,}$ ... from $A$ and all $i \in$I, $j \in \mathcal{J}$.

Or in any category of algebraic systems the isomorphism is the homeomorphism which is bijection. The isomorphism of algebraic system on itself is called automorphism. The homeomorphism is a morphism in category of algebraic systems. Homeomorphism is the mapping of algebraic system A keeping the main operations and the main relations in it. The *morphism* of category is a term, for designation of the elements of any category playing a role of mappings of sets each other, gomoyemorfizm of groups, rings, algebras of continuous mappings of topological spaces, etc.

5. *Variety* is the geometrical object which locally have a structure (positive, smooth, homological or other) numerical space of $R^n$ or other vector space. This fundamental idea matematiks the concepts of the line and a surface specifying and generalizing on any number of measurements.

6. *The stream*, dynamic system, with continuous time is the dynamic system determined by action of additive group of real numbers $R$ (or additive semi-group of non-negative real numbers) on some phase space of $X$. Otherwise, to each $t \in R$ some transformation $\varphi_t : X \to$ X, and

$$\varphi_0(x) = x, \qquad \varphi_{a+b}(x) = \varphi_a(\varphi_b(x)). \qquad (2.1.3)$$

7. *The stream of the vector field* $\alpha$ through a surface $\sigma M$ is expressed to within the sign by superficial integral

$$\iint_{\sigma M} \left(a_x d_y d_z + a_y d_x d_z + a_z d_x d_y\right), \qquad (2.1.4)$$

where $n$ is a single vector of a normal to a surface σM. For example, for the vector field of speeds the stream of the vector field is equal.

8. *Shift.* As shift in the theory of dynamic systems it is understood, affine transformation on itself the $n$ planes of measured space at which each point is displaced in the direction of an axis $O_x$ on the distance proportional to her ordinate or positive coordinate.

In the Decart's system of coordinates the shift on the plane is set by ratios:

$$x^1 = x + k^z, y^1 = y, z^1 = z, k \neq 0. \qquad (2.1.5)$$

At shift of 3-dimensional space volumes and orientation remain.

*Shift the operator* is the $T_t$ operator depending on parameter $t$ and acting on some set ϕ map φ: $A \rightarrow M$ (where $A$ is an abeleva subgroup, $M$ is a set) on a formula

$$T_t \varphi(\cdot) = \varphi(\cdot + \mathrm{e}). \qquad (2.1.6)$$

9. *A homothety* – the transformation of Euclidean space of rather some point of $O$ putting in compliance to each point the $M$ point of $M^1$ lying on direct $OM$ on a ratio

$$OM^1 = k\, OM,$$

where $k$ is number, relative, other than zero, is called homothety coefficient. The point 0 is called the center of a homothety.

10. *The endomorphism* of algebraic system is the map of algebraic system $A$ to itself coordinated with its structure i.e. if $A$ algebraic system which signature consists of a set $\Omega_F$ of symbols of operations and set $\Omega_P$ of symbols of a predicate that endomorphism φ: $A \rightarrow A$ has to meet the following two conditions:

1) $\varphi(a_{1,} \ldots, a_{n,}\varpi) = \varphi(a_1) \ldots \varphi(a_n)\varpi$ for any *n*-th operation $\varpi\epsilon\Omega_F$ and of any sequence of elements $a_{1,} \ldots, a_{n,}$ systems *A*;

2) $P(a_{1,} \ldots, a_n) \Longrightarrow P\left(a_1 \ \varphi, \ldots, aa_{1,_n}\varphi\right)$ for any *n* a local predicate of $P \leftarrow \Omega_P$ and *A*.

The dynamic systems by the form of functions from time are divided into dynamic systems continuous and discrete. At the description of discrete dynamic systems (further *DS*) analysis questions become simpler a little because the map generating *DS* with discrete time can quite often be set obviously, by means of some formulas. The systems with continuous time are set, usually infinite dimension and restoration of dynamics according to such description of *DS* includes the process representing an integration analog.

In a continuous case, designating $\sigma/_{\sigma x_i}$ basic vector fields which compare to each point of *i* is *a vector of standard basis of* $R^n$, it is possible to provide each vector field locally in a look

$$\sum_{i=1}^{n} f_i \ (x_{1\,,} \ldots, x_{n,}) \frac{\partial}{\partial x_i} \quad .$$

If the initial point of $x^0$ is set by coordinates of $x_1^0,\ldots,x_n^0$, then to the task of the movement of this point is defined as a result of the solution of system of the ordinary differential equations of the first order

$$\frac{d_{xi}}{d_t} = f_i\left(x_{1\,,} \ldots, x_{n,}\right), \qquad (2.1.7)$$

with entry conditions $x_i(0) = x_1^0, i = \overline{1,n}$ .

From the theory of the ordinary differential equations it is known that if functions $f_i$ are continuous and differentiated the decision, (2.1.7) exists, only and smoothly from entry conditions.

# CHAPTER 3. SYNERGETIC SYSTEMS

## 3.1. Self-organization in natural systems

### 3.1.1. Mathematics and mechanics

The simplest abstract examples of systems where signs of self-organization are observed and randomnesses are examples of the logistic equation describing growth of populations and also the Henon's maps, the "horseshoes" type ("Smale's horseshoe") and "baker".

The logistic equation or the equation of growth of population, has an appearance:

$$x_{n+1} = ax_n - bx_n^2, \tag{3.1.1}$$

where $a$ and $b$ – parameters.

The nonlinear model (3.1.1) is presented in the dimensionless form

$$x_{n+1} = \lambda x_n(1 - x_n) = f(x_n). \tag{3.1.2}$$

At parameter $\lambda > 1$ there are two points of balance:

$$x_0 = 0 \text{ и } x_0 = \frac{\lambda - 1}{\lambda},$$

at the same time, at $1 < \lambda < 3$ the beginning of coordinates $x = 0$, the unstable point $(f' = \partial f / \partial x_n = \lambda > 1)$, and the second point $(f' = 2 - \lambda)$ of rest is stable.

Further, at value $\lambda = 3$ the inclination $x_0 = (\lambda - 1)\lambda$ exceeds unit, and both points $|f'| = \left| \partial f / \partial x_n \right| > 1$, $f' = 2 - \lambda$ of balance are unstable.

At there $\lambda = 3$ is unstable a stationary decision, but the stable two-periodic cycle appears. At further increase $\lambda$ the two-periodic orbit becomes unstable and there is a cycle with the period 4 which owing to bifurcation doubles the period to 8 at great values $\lambda$.

This process of doubling of the period continues to value $\lambda_\infty = 3{,}56994...$. At the same time near this value the sequence of the values of parameter $\lambda$, doubling the period submits to the exact law Feigenbaum's

$$p = \frac{\lambda_{n+1} - \lambda_n}{\lambda_n - \lambda_{n-1}} \to 4{,}66920166.... \quad (3.1.3)$$

The irrational number of $p$=4.66920166 … is called Feigenbaum's number by name the American physicist Mitchell Feigenbaum who for the first time in 1976 has found these properties of the considered equation (mapping).

At the values $\lambda$ exceeding $\lambda_\infty$ in the systems described by the equations of type (3.1.2) there can be chaotic iterations (oscillations). But at an interval $\lambda_\infty < \lambda < 4$ there are also some intervals $\Delta\lambda$ for which there are periodic orbits (regular oscillations).

Value of mapping (3.1.2) not only that it is an example of the simplest system showing formation of chaos but also that on this example are found universal property of the equation of the periods of classes of one-dimensional differential models of dynamic processes.

The equations of the period and Feigenbaum (3.1.3) relation, are characteristic to many mappings $x_{n+1} = f(x_n)$ of a high order (above the first) and are found in many scientific experiments.

The following example this is Henon's map described by the two-dimensional differential system of the equations offered by the French astronomer Henon

$$x_{n+1} = 1 - ax_n^2 + y_{n,}$$

$$y_{n+1} = bx_{n,} \;, \quad (3.1.4)$$

where $a$ and $b$ mapping parameters.

At $|b| < 1$ mapping (3.1.4) reduces the areas in the plane *x0y*. Besides, extends mappings and squeezes areas on the phase plane.

As a result of these operations on stretching, compression, a bend and folding of areas of phase space the areas reminding a horseshoe turn out. Therefore the Henon's map belongs to "horseshoe" maps, sometimes call also Smale's horseshoe. Consecutive iterations of such "horseshoe" maps lead to formation of difficult movements and chaos.

The attractor (the attracting variety) to which aspires a mapping point, presents the work of one-dimensional variety on a Cantor set, i.e. has fractal structure.

The researches conducted at values $b = 0.3$; $a -$ var. Reveal the following points of bifurcation:

$$a_0 = -(1-b)^2/4 = -0.1225;$$

$$a_1 = \frac{3}{4}(1-b)^2 \cong 0.3675;$$

$$a_2 \cong 1.06;$$

$$a_3 \approx 0.55.$$

At $a < a_{0,}$ or $a > a_3$ points always go to infinity, at these and the attractor doesn't exist.

At $a_0 < a < a_1$ an attractor it is a stable invariant point. When $a > a_1$ the attractor is periodic a set from $q$ of points, similar to a limit cycle. With growth of the value $q$ grows and strives for infinity at $a_2 \approx 1.06$. At $a_2 < a < a_3$ an attractor difficult, but the question of "strangeness" (randomness) of this attractor raises certain doubts recently and remains open.

Many researchers believe that "horseshoe" map plays a fundamental role in the majority of models of the chaotic dynamic systems based on the differential and differential equations. For example the system describing the movements of dot weight on a spring with friction and to conditions of absolutely elastic blow about the limiter has chaotic to the " loudspeaker Smale's horseshoe".

Transformation of "baker" also represents an example of chaotic dynamics.

Transformation of "baker" is a transformation of the plane on itself which stretches the rectangular platform in one direction, squeezes her in other direction, cuts in half and places one half over another. This transformation similar to "horseshoe" transformation. Repeated iterations of this transformation turn an initial set of points into fractal structure. Transformation is called on similarity to the operations made by the baker kneading a piece of the test: roll out, extension, cutting and rearrangement.

Transformations of "baker" it is set by the system of two differential equations:

$$y_{n+1} = \begin{cases} 2y_{n,} npu\, y_n < \frac{1}{2}, \\ 2(y_n - \frac{1}{2}), npu \quad y_n > \frac{1}{2}, \end{cases} \quad x_{n=1} = \begin{cases} \lambda_a x_n, npu \quad y_n < \frac{1}{2}, \\ \frac{1}{2} + \lambda_b x_n, npu \quad y_n > \frac{1}{2}; \end{cases} \quad (3.1.5)$$

where $\lambda_a, \lambda_b$ are transformation parameters.

As show many researches of this transformation and here too there are chaotic acyclic movements reminding on complexity casual, but as the equations determined, they are called the determined chaos.

The following example of chaotic dynamics is observed in the problems of so-called "billiard" type modeling the systems of statistical mechanics.
The billiards on the plane is the system describing the movements on inertia of material bodies (spheres) in limited area under the law "the hade is equal to the angle of reflection". From the mathematical point of view "billiards" represents ordinary billiards, but with any form of a table and without billiard pockets. By the researches published in many references it is shown that the system even from two spheres, depending on a form of border can possess to properties of chaotic dynamics. Thus, instability or unpredictability of trajectories of system of elastic spheres. On the basis of researches of "billiard" systems the result about "convergence to Brownian motion of

behavior of purely determined system has been received that the chaos birth in determined dynamic systems was strict confirmation.

Generalization of "billiard" systems are billiards which borders aren't motionless, and change under any certain law. For example, the task about dynamics of a sphere can serve in billiards where the border changes as model of problems of nonequilibrium statistical mechanics. Many tasks of "billiard" systems are set and solved by school of the Soviet mathematicians Ya.G. Sinay.

The simplest example of the mechanical systems having difficult dynamics are pendulums. Pendular systems show significantly the "nonlinear" phenomena (multistability, bifurcations, chaos).

The movement of the simple pendulum with friction can become chaotic at excitement by harmoniously changing force of sufficient amplitude.

The dimensionless equation of the movement of the pendulum in coordinates of an angle of rotation is presented in the following form:

$$\ddot{x} + a\dot{x} + \sin x = b \cdot \cos x \cdot \cos \omega t . \qquad (3.1.6)$$

Chaotic oscillations meet in the neighborhood of own frequency at small oscillations.

The chaotic movements are observed under certain conditions and in dynamics of a three-sedate gyroscope with nonlinear damping at the external harmonious influence caused by vertical vibration of the basis, chaos is observed also in the mechanical systems of cores and beams, plates, shock systems and chains of the oscillators which are consistently connected by elastic communications.

### 3.1.2. Physics.

Many systems from hydrodynamics show examples of chaotic movements.

There are five types of tasks with liquids in which the chaotic movements are observed:

systems with the closed currents: convection Rayleigh – Benard, the current Taylor - Couette a boundary cylinders;

open currents: a current in a pipe, interfaces, streams;

liquid particles: the proceeding crane;

waves on the surface of liquid: gravitational superficial waves;

the reacting liquids: the mixed tank of the chemical reactor.

The main reason for indefatigable interest in chaotic dynamics in liquids this is possibilities of disclosure of mechanisms of formation of turbulence. Knowledge of laws and mechanisms of turbulence will allow to develop further methods of controlling of this very important in the applied phenomena.

**Thermal convection of Rayleigh - Benard.** Temperature gradient in the liquid which is in the field of inclination creates force of buoyancy which causes vortex instability and brings to chaotic and to whirls. Formation of so-called cells of Benard in the closed rectangular volume at uniform heating of the lower plate is the most studied. Experimental a research of thermal convection of Rayleigh - Benard in the closed volume was shown that harbingers of the chaotic movement are the sequences of doubling of the period. These experiments have been made with various liquids in which are found as transition to chaos through quasiperiodic oscillations, and the alternated chaos.

**System of Taylor - Couette.** Classical hydromechanical system where preturbulent chaos is found, the current between two rotating cylinders is. In such system before establishment of chaotic noise quasiperiodic oscillations are observed. Some researches have shown possibilities of controlling of emergence or suppression of chaos by means of change of speed of rotation of the internal cylinder.

Historically the first example of the determined systems where was, has found the chaotic movements there was an example of system of Lorenz. Thereby in 1963 the meteorologist E. Lorenz for the first time has experimentally confirmed theoretical opening of H. Poincare (1892) that in some mechanical systems described by the determined equations there can be chaotic oscillations arising in Lorenz's system in the subsequent have been called to "strange attractors" of Lorenz.

The equations describing Lorenz's system have an appearance:

$$\dot{x} = \sigma(y - x),$$
$$\dot{y} = \rho x - y - xz, \qquad (3.1.7)$$
$$\dot{z} = xy - \beta z,$$

where $x, y, z, \in R$ are variable conditions of system, $x$ is proportional to amplitude of speed of the movement, and variables $y, z$ reflect distribution of temperature in a convective ring, σ, ρ are the positive parameters connected with Prandtl's and Rayleigh's numbers, $\beta > 0$ are the parameter characterizing system geometry.

In many researches of system of Lorenz (3.1.7) believe σ = 10, $\beta = 8/3$, and parameter ρ is varies.

At $\rho$<1.0 in Lorenz's system there is the only special point (*SP*) at the beginning of coordinates $SP_1(0,0,0)$ like "stable knot". Further at ρ = 1.0 there is a bifurcation to formation of two new special points of $SP_2$ and $SP_3$ type "stable knot":

$SP_2(a, a, p-1)$; $SP_3(-a, -a, p-1)$, where $a = \left[8/3\,(p-1)\right]^{1/2}$.

At ρ =1.345 in Lorenz's system there is a bifurcation of change of types of special points $SP_2$ , $SP_3$ , namely these special points turn into special points like "stable a saddle focus". Further at ρ =24.74 there is Hopf bifurcation (Poincare - Andronov – Hopf) when couples of eigenvalues in $SP_2$, $SP_3$ become purely imaginary and at $\rho$ >24.74 these special points become like "unstable a saddle - focus". At the same time in Lorenz's system there is "a strange attractor" of Lorenz when in limited area of three-dimensional space, around "unstable a saddle - focuses" $SP_2$, $SP_3$ there are chaotic oscillations covering a saddle point $SP_1$.

One of examples of systems where there are chaotic movements in the form of turbulence in liquid is the system of the third order described by the Lengford's system:

$$\dot{x} = (2a-1)x - y + xz,$$
$$\dot{y} = x + (2a-1)y + yz, \qquad (3.1.8)$$
$$\dot{z} = -az - (x^2 + y^2 + z^2).$$

In system (3.1.8) at $a = 1/2$ , there is Hopf bifurcation to formation of stable oscillation with parameters $a_0 = 1/2$ and $T_0 = 2\pi$ .

**Plasma.** It is known that plasma consists of gas or liquid which atoms partially or are completely deprived of the electron shell, i.e. are ionized. In plasma a set of various not stability are observed. The research of instability of plasma and control of a state and the movements of plasma are very important for a solution of the problem of the operated thermonuclear synthesis.

One of plasma models with chaotic behavior for a research of controllability of plasma and suppression of chaos is represented the three-dimensional equation of a look:

$$\dot{x} = -ax - b_1(x+z)y^2,$$
$$\dot{y} = -a_0 y + b_2(x^2 - z^2)y, \qquad (3.1.9)$$
$$\dot{z} = -a_0 z + b_3(x+z)y^2,$$

where $x, y, z$ are of amplitude of the reflected wave, waves a rating and a direct wave, $a, a_0, b_1, b_2, b_3$ are parameters.

Differentiation is made on spatial coordinate along the direction of distribution of waves. At $a_0 = 1$ in system (3.1.9) the chaotic behavior is observed.

In the last decades many researchers, dealing with issues of **geology, geophysics, geoecology** find many phenomena in these areas characteristic to synergetic systems namely effects of self-organization are spontaneous education spatial and spatially – temporary structures through various bifurcations and accidents in these systems. So, a lithosphere and its separate parts are open dynamic nonlinear systems which exchange among themselves and with the environment substance and energy.

The tectonic environment is various, it not continuous, and contains the emptiness (a time, cracks, etc.) filled with fluids (liquids and gases). There are manifestations of lamination and a blokness of different scales. Formation folds are characterized by spatial rhythm of different orders. Wavy forms of surfaces of breaks are noted.

It is known that Earth as a space body develops in time. Constant receipt of energy in her from the outside, is the major sources which radioactive decay, sunlight and tidal processes are the reason of temporary development of Earth.

Energy inflow from the outside creates conditions for Earth substance heatmass transfer, the causing movement of substance of surfaces and in a planet subsoil. The lamination and a blokness of different scales with hierarchical structure of the sizes of blocks is shown. The law of repeatability of earthquakes of Guttenberg and Richter acts as perturbation of hierarchical distribution of blocks of a lithosphere by the sizes.

Thus, the geophysical environment represents the open self-organized dynamic system, and geophysical processes, including seismic, are nonlinear processes.

Process of self-organization is connected with emergence in the active environment, for example, a seismoactive layer, the localized dissipative structures which are characterized by not stationarity, an impulsness, complexity and degradation.

In seismic processes such typical structures of self-organization as are observed: spiral waves or the focused triangles; whirlwinds or the focused polygons; seismic "paths" which happen unidirectional ("chains") and forward and returnable ("pendulums"), ring seismicity ("calm zones" and seismic "gaps") and seismic swarms.

The chaotic movements are possible also in many elements of physics of a solid body this in Gunn's oscillators, systems with tunnel diodes and dipolar domen.

### 3.1.3. Chemistry

Chaotic oscillations in chemical reactions have been for the first time studied in 60th and 70th the 20th century. The most known model where has found the chaotic movements the **brusselyator model** was. This model describes distribution on space and change of reagents of rather narrow class of chemical reactions over time. The model of a brusselyator is in many respects studied by the Brusseles school of thermodynamics of the Nobel laureate I. Prigogin.

The model of a brusselyator has been offered by I. Prigogin and R. Lefebvre in 1968. For a research of nonequilibrium dissipative structures of chemical systems. This model clearly shows how the nonequilibrium system can become unstable and pass to oscillating motions.

In a dimensionless look the equation of a brusselyator has an appearance

$$\dot{x} = a-(b+1)x+x^2y, \quad \dot{y}=bx-x^2y, \tag{3.1.10}$$

where $a$ and $b$ are parameters.

At $b=a^2+1$ in system (3.1.10) there is Hopf bifurcation to formation of periodic oscillations at $b > a^2 + 1$.

The following example of chemical systems with synergetic behavior is Rössler's system which is described by the equations

$$\dot{x} = -y - z, \quad \dot{y} = x + ay, \quad \dot{z} = bx - cz + xz, \tag{3.1.11}$$

where *a, b, c* positive parameters. In this system two special points of $x_0 = y_0 = z_0 = 0$, and $x_0 = c - ab$, $y_0 = b - {}^{c}/_{a}$, $z_0 = {}^{c}/_{a} - b = -y_0$.

Here two options of chaotic movements this is spiral and screw character are possible.

**Belousov-Zhabotinsky system.** The model offered by Belousov (1959) and Zhabotinsky (1964) describes chemical reaction of catalytic oxidation of melanovy acidity $CH_2\,(COOH)_2$. Reaction happens in water solution and is easily carried out in a flask at simple mixing of some reagents in certain concentration. The oscillatory behavior in system can be revealed on change of concentration of $CE^{4+}$ causing change of coloring of solution from colourless to yellow and brighter colors.

The option of reaction of Belousov-Zhabotinsky on model of Fild-Kyoresh-Noyes is described by the equations

$$\dot{x} = k_1 ay + k_2 ay - k_3 xy - 2k_4 x^2,\ \dot{y} = -k_1 ay - k_3 xy + {}^{1}/_{2}\, k_5 bz, \quad (3.1.12)$$

$$\dot{z} = 2k_2 ax - k_5 bz,$$

where $k_1 = 1.28$; $k_2 = 8.0$; $k_3 = 8.0 * 10^5$; $k_4 = 2.0 * 10^3$; $k_5 = 1.0$; $a$=0.06; $b$=0.02; $0.5<f<2.4$.

In Belousov-Zhabotinsky reaction various oscillations, including chaotic are found.

In system (3.1.12) depending on f two or three special points, one of which the beginning of coordinates (0,0,0). Chaotic oscillations happen at $0.9208 < f < 1.0808$, bifurcation at $f = 0.9208$ and $f = 1.0808$.

In applied aspect there is interest in controlling of oscillations in Belousov-Zhabotinsky system, as for the purpose of suppression of chaotic oscillations, and excitement of the oscillatory or chaotic mode.

### 3.1.4. Biology

From all natural objects living beings, undoubtedly, and functionally and morphologically are the most high-organized. Living beings are historical structures capable to hold in remembrance and a form, i.e. information on historical evolution of these beings. They function far from balance. The organism receives continuously energy and substance and also information as memory unit from the world around.

At the cellular level of living beings it is also observed nonequilibrium, for example on the maintenance of ions of sodium and potassium in a cage and in the noncellular environment.

Researches establish connection between physical and chemical forms of the organization of structures and biological orderliness. One of examples of such researches is a development of an amoeba like Dictyostelium discoidenum. It is revealed that in essence development of this live organism comes down to the transitional phenomenon similar to Belousov-Zhabotinsky reaction, and noting transition from life, monocelled to a multicellular stage of development. Life cycle of these organisms is described in many references on synergetrics.

Evolutionary development it is also possible to consider as education all of new and new macroscopic structures, as a result of survival of the most adapted types of biomolecules and organisms in general. Believe that biomolecula avtokatalitic at the expense of a cyclic catalysis in hyper cycles breed. Researches show that such selection in combination with mutations can lead to evolutionary development.

### 3.1.5. Ecology

The nonequilibrium, chaos and formation of communities of populations of a plant and animal life are characteristic ecology. For example, formation of endemic zones of vegetation. Lead to macroscopic changes also environmental pollution which can lead to disappearance of the whole species of an animal and flora.

The simplest model of an ecosystem is the "predator-prey" model offered A. Lotka and V. Volterra in the twenties of the last century it is also described by the equations of the second order:

$$\dot{x} = \alpha x - \beta y, \quad \dot{y} = k\beta xy - my, \qquad (3.1.13)$$

where *x=x(t), y=y(t)* are the number of populations according to the preyes and predators, α and β are the multizian's and trophic constant preyes, *k* is efficiency of processing of biomass of the prey in biomass of a predator, m is coefficient of natural mortality of a predator.

More difficult model of ecological system describing system with development of populations of a flour bug of Tribolium is represented so-called *LPA* model:

$$\begin{aligned} L_{n+1} &= bA_n \exp(-a_1 L_n - a_2 A_n), \\ P_{n+1} &= L_n(1 - C_1), \\ A_{n+1} &= P_n \exp(-a_3 A_n) + A_n(1 - C_2), \end{aligned} \qquad (3.1.14)$$

where *L* and *P* are according to number of the raised and not raised larvae, *A* is number of individuals, ready to reproduction, $a_1, a_2, a_3, b, c_1, c_2$ are the parameters.

Researches note that in system (3.1.14) are observed: the established states, periodic, quasiperiodic, and chaotic oscillations at various values of parameters. Also possibilities of control of the movements of this system for maintenance of a certain quantity of insects are investigated.

In conclusion of this section it is possible to note that self-organization is characteristic to very many phenomena and natural systems and studying of the nature of regularities of these phenomena and systems allows to predict and to operate to some extent these systems for the purpose of parrying of undesirable effects and strengthening of positive consequences of self-organization.

## 3.2. Self-organization in technical systems

### 3.2.1. Mechanical systems

In many technical systems there are different oscillations, in particular periodic and chaotic which have to be predicted and in necessary cases are operated.

These oscillations arise as under the terms of operation of mechanisms and systems, and an undesirable image, owing to various oscillations, hindrances, imbalance of mechanisms and the rotating parts, vibrations in designs, etc.

Classical example of self-oscillations systems and not only electric, but also a wide class of systems, in particular mechanical systems with friction, is the system described by the equation Van der Pol

$$\ddot{x} - \mu\dot{x}(1 - \beta x^2) + \omega_0^2 x = 0, \qquad (3.2.1)$$

where $x \in R$ is the coordinate of the movement, $\mu$, $\beta$, $\omega_0$ are parameters.

At system (3.2.1) at small $\mu$ there are quasiperiodic self-oscillations, and at big $\mu$ self-oscillations have relaxation (explosive) character.

The equation Van der Pol is the most studied example of oscillatory systems.

The following example of mechanical systems where more difficult movements are possible is the example of the mechanical rotator with the moment of inertia of $J$ and attenuation $\mu$.

The equation of the rotator has an appearance

$$J\dot{\omega} + \mu\omega = \mu\omega_0 + F(\varphi)\sum_{n=-\infty}^{\infty}\delta(t - n\tau), \qquad (3.2.2)$$

where $\varphi$ is an angle of rotation, $\delta(t\text{-}n\tau)$ is delta function, such that at $n\tau - \varepsilon < t < n\tau + \varepsilon,\ \varepsilon \ll 1$:

$$J(\omega^+ - \omega^-) = F(\varphi(n\tau)), \quad F = F_0 sin\varphi. \qquad (3.2.3)$$

In system (3.2.2) as show researches at certain values of parameters there is a strange attractor.

**Flutter** is the aeroelastic oscillations. The flutter is the oscillations caused by a current of liquid over an elastic plate. In the systems described by the flutter phenomenon in an avia and space equipment the acyclic and chaotic movements are observed.

The system of "flutter" is described by the equations

$$\ddot{x} + \mu\dot{x} + [1 - a + x^2 + 4y^2]x - Ay = 0,\ \ddot{y} + \mu\dot{y} + 4[4 - a + x^2 + 4y^2]y - Ax = 0, \qquad (3.2.4)$$

where $a$ is the quantity of tension in the plate plane, $A$ is quantity proportional to the dynamic pressure of a stream before a plate, μ – parameter.

The **mechanical control** system described by the equations

$$m\ddot{x} + \mu\dot{x} + F(x) = -z, \quad \dot{z} + az = b_1\left[x - x_f(t)\right] + b_2\dot{x}, \qquad (3.2.5)$$

where $z$ is a variable of feedback, $b_1, b_2$ are feedback coefficients by situation and on the speed, $x_f(t)$ is nonlinear influence in system.

In the considered system both periodic oscillations on a limit cycle, and chaotic oscillations, in particular at are possible $F(x) = x(x^2 - 1)(x^2 - c)$.

### 3.2.2. Electric and electronic systems

In the last decade in literature has appeared many publications concerning questions of self-organization and emergence of various difficult movements in electric and electronic systems. Are found both periodic, quasiperiodic, and the chaotic movements in the systems of the considered type.

It is known that in electrical equipment conclusions concerning a possibility of emergence of difficult chaotic movements, the analysis of bifurcations and determination of parameters of oscillations are rather simply feasible. Therefore many known "classical" examples of chaotic oscillations and movements (in the systems of Chua, Lorenz, Rössler, etc.) are almost embodied in electrotechnical systems.

Classical example of electric systems where chaotic oscillations are observed, the circuit of Chua's described by the system of three equations is:

$$\dot{x} = p(y - f(x)), \quad \dot{y} = x - y + z, \quad \dot{z} = -qy, \qquad (3.2.6)$$

where $p$ and $q$ are parameters, $f(x)$ is the nonlinear function determined by a ratio

$$f(x) = M_1 x + 0.5(M_1 - M_0)(|x+1| - |x-1|),$$

where $M_1, M_0$ are parameters.

At certain values of parameters, in system (3.2.6) chaotic oscillations are possible, for example at $p = 9$; $q = 14.3$; $M_1 = -6/7$; $M_0 = 5/7$.

At the same time,

$$f(x) = \begin{cases} -\frac{6}{7}x + \frac{11}{7}, \ x < -1\,, \\ -\frac{17}{7}x, \ -1 \le x \le 1, \\ -\frac{6}{7}x - \frac{11}{7}, \ x > 1. \end{cases} \quad (3.2.7)$$

There are three special points:

$SP_1$ (0,0,0); $SP_{2,3}$ (±11/6,0 ,± 11/6).

The circuit of Chua's, namely the chaotic modes of this scheme are used in various devices, for example as the generator of the chaotic controlling director of a signal in the device of chaotic pulse-width modulation for elimination of the alternating dark and light strips seen on fluorescent lamps.

The **electromechanical converter** which is system, two-piece is the nonlinear electric part described by the equation of Duffing's and a mechanical part, which is linear oscillatory system is also characterized by possibility of chaotic oscillations.

The system of the electromechanical converter is described by the equations:

$$L\ddot{q} + R\dot{q} + \frac{q}{C_0} + \alpha q^3 + LH\dot{z} = \mathrm{a}\cos\omega t, \quad m\ddot{z} + \rho\dot{z} + kz - lH\dot{q} = 0, \quad (3.2.8)$$

where $L$ and $R$ are respectively inductance and active resistance in an electric part; $a$, $\omega$ are amplitude and frequency of external harmonious tension; $l$ is the extent of the site of interaction of magnetic field tension of $H$ with two mobile cores on which the body with a mass of $m$ fastens; $k$ is coefficient of elasticity of a spring; ρ is coefficient of viscous friction; $q$ is a charge on condenser facings, tension on which facings depends not linearly on the quantity of this charge $U_c = {}^{q}/_{C_0} + \alpha q^3$; $C_0$ is linear part of the capacitor characteristic of the condenser; α is the parameter defining nonlinearity of the characteristic of the condenser; $\dot{q}$ is current in an electric part; $z$ is movement coordinate in a mechanical part of system.

Many devices of electronics and computer facilities are constructed on semiconductor elements (devices) of various extent of integration, since discrete components are diodes and transistors, to modern integrated chips, *IMS* and *GBSI.*

Therefore researches of various phenomena of self-organization in semiconductor elements is very important for development of modern electronic devices of varying complexity, since cell phones and scientific or medical devices to supercomputers and computer networks.

The main reasons for not stability and nonequilibrium in semiconductors are processes of generation and a recombination of carriers of a charge or so-called *GD*-processes which bring in some semiconductor devices. For example in tunnel diodes, Gunn's diodes, avalanche-flying diodes (*ADC*), multilayered devices like tiristor, *p-n-p-n* diodes, *p-i-n* diodes, etc. to significantly nonlinear volt-ampere characteristics like *NDC* (negative and differential conductivity) of two types of *N* type and *S* type.

Chaos in semiconductors can arise in various ways: or because of reactivity in an external circuit, or owing to own instability of an element with *NDC*.

### 3.2.3. Lasers

Process of self-organization is characteristic also to lasers which let out coherent light radiations under nonequilibrium conditions. It is known that lasers represent special type of lamps which consist of a crystal core (solid-state lasers) or the glass tube filled with gas (gas lasers).

If to excite or "pump up" from the outside atoms of which the working body of the laser consists then they let out light waves. At the low power of "rating" these waves aren't correlated, as in radiation, by the let-out usual lighting lamp. At the big power of the "rating" equal to some critical, atoms let out purely sinusoidal light wave, i.e. separate atoms work in strictly correlated way or otherwise self-organizations. At excess of power of a rating of the second critical value, the laser begins to let out intensive and short impulses periodically. Under various conditions emission of light can become chaotic or turbulent, i.e. chaotic.

In the system of two lasers becomes feasible the bistable optical systems applicable in the optical memorable and logic devices of optical computers. The bistability, and also chaos, arises and when the laser is connected with the so-called sated absorber, i.e. material which coefficient of transmission becomes very big at high intensity of light. In modern use of laser systems problems of controlling of laser radiations for the purpose of suppression of chaotic (multiannual) behavior of lasers by means of feedback are solved with delay and also the opened (program) control.

**3.2.4.** Some examples of the phenomena of self-organization are known in **computing systems**: in the system of parallel calculations of computers; at recognition of images by means of the computer; at any creation of reliable systems, including on the basis of computers from unreliable (more precisely from insufficiently unreliable) elements.

**3.2.5.** The increasing application is received by some positive characteristics of chaotic movements and oscillations in modern **communication systems.**

These advantages of dynamic chaos are caused it by the following characteristics:

this is a wide-stripness, chaotic signals have a continuous wide range;

this is the complexity of structure of chaotic signals allowing to create absolutely different signals at minor changes of entry conditions. Such property of chaotic signals is possible use in cryptography;

this is orthogonality (non correlatedness) of signals of chaotic generators caused by irregularity of chaotic signals. This property is applicable in the multiuser communication systems when the same range of frequencies is used by several users at the same time.

Dynamic chaos receives application in such directions of communication systems as: synchronization of the receiver and transmitter, masking and recovery of messages filtration of noise, development of algorithms of coding, decoding, etc.

The system where Lorenz's model for transmission of messages is used is known. In this system the transmitter is described by the equations:

$$\dot{x} = \mathrm{G}(\mathrm{y} - \mathrm{x}), \quad \dot{y} = \rho x - y - 20xz, \quad \dot{z} = 5xy - b_1 z, \tag{3.2.9}$$

where $x, y, z$ are the system coordinates corresponding to tension at the exits of amplifiers, at the same time in system $G$ = 16, ρ=45.6, $b_1$ =4.0 .

The equations of the receiver of system of the message are also described by Lorenz's equations of a look (3.2.9) where respectively variable states $x_s, y_s, z_s$, parameters same, as in (3.2.9).

The system of the receiver is the asymptotic observer for the transmitter (3.2.9):

$$\lim_{t\to\infty}\|e\| = \lim_{t\to\infty}\left\|\begin{matrix} x - x_s \\ y - y_s \\ z - z_s \end{matrix}\right\| \to 0.$$

By transfer of a binary signal of the message, the coefficient of $b_1$ of the transmitter changes, accepting $b_1$ = 4.4 value that corresponds "1" whereas the reference value of $b_1$ = 4.0 means "0". At change of $b_1$ from 4.0 to 4.4 mismatch signal level on coordinate $x$, $e_x = x - x_s$, as for the $b_2$ = 4.0 receiver sharply increases in the system of the receiver. Is defined by averaging of $e_x^2(t)$ what signal "1" or "0" has been transferred.

Other systems with chaotic dynamics, in particular the operated Chua's system also are applied to transmission of messages with use of chaotic generators. Possibilities of use of systems with chaos for communication systems are very wide and perspective.

**3.2.6.** Recently believe that tempting prospects are offered in use of chaotic dynamics for the systems of **storage and coding of information**. These systems of information processing are supposed to be constructed on the basis of so-called "chaotic processes". A number of developments in this direction are patented in the USA and the Russian Federation.

**3.2.7.** Broad application of chaos and the ideas of self-organization receive in **chemical industry** when reactions like Belousov-Zhabotinsky or Rössler, and also possibilities of chaotic dynamics. So chaos receives application in processes of chaotic hashing of liquids and loose substances. High-quality hashing is important process in combustion chambers, heat exchangers, mixer reactors of continuous action and other productions.

**3.2.8.** Very perspective applications of the ideas of self-organization and chaotic dynamics are shown many researchers and developers of the **space equipment**, in particular for control.

For example, the problem of forecasting of the chaotic movement and control by him in a gyrostat is solved. It is known that a gyrostat this solid body having three rotary degrees of freedom in which there are one or several flywheels.

As in general, generally the gyrostat is described by the nonlinear equations and in practical applications is opened for the external environment and is subject casual powerful indignation, in the system of a gyrostat the chaotic movements are possible, control of which are very important in the missile and space equipment.

So, the movement of the satellite at simultaneous influence of gravitational and magnetic fields of Earth is investigated. The satellite having own constant magnetic field is considered. The equation of the movement of such satellite takes a form:

$$k\ddot{x} + k\dot{x} + 3\omega_c^2(B - A)\sin x\ \cos x + \mu_m \rho I r^{-3}(2\sin x \sin \omega_c t + \cos x \cos \omega_c t) = M_c(t), \tag{3.2.10}$$

where *x=x(t)* a satellite libration corner in the orbit plane; $\omega_c$ is the angular speed of the movement of the satellite on an orbit; *k* is coefficient of damping of the satellite; *A, B* are the main moments inertia of the satellite (*B >A);* $\mu_m$ is a magnetic constant; *r* and $\rho$ are the radius and an inclination of an orbit; *I* is quantity of the magnetic moment of the satellite; $M_c(t)$ is value of the operating moment.

In system (3.2.10) in some area of parameters the angular movement of the satellite at $M_c \equiv 0$ has chaotic character.

In this case of course suppression of undesirable chaotic oscillations is necessary. Such problem is solved with the help of feedback with an exit and with a derivative.

Some areas and examples of applications of self-organization and chaos in technical systems are shown in this section, but the number of such applications quickly grows eventually and many potential opportunities of such applications aren't exhausted.

# CHAPTER 4. THEORY OF ROUGHNESS AND BIFURCATIONS OF SYSTEMS

## 4.1. Property of roughness of dynamic systems

The property of roughness is one of fundamental properties of dynamic systems. In modern foreign literature on dynamic systems the property of roughness is called property of structural stability. We in this work will adhere to the term "roughness".

Already the most general definition of the concept "roughness of dynamic systems" (further DS) is the smooth DS having property: for any ε > 0 there will be such $S > 0$ that at any perturbation of DS remote from her in $C^1$ a metrics no more than on $S$, there is "homeomorphism" of phase space (or variety) which shifts points of no perturbation system in the corresponding trajectories of the perturbation system.

The definition given above is definition of a concept of roughness of DS "in narrow", i.e. roughness in a certain topological space. In wider concept the roughness assumes properties preservation of some property of system at certain small perturbations.

The concept of roughness has been introduced by outstanding soviet scientists A.A. Andronov and L.S. Pontryagin for the first time.

The concept of roughness with the requirement of "homeomorphism" of the perturbation and no perturbation systems to ε for rough DS carries the name of roughness according to Andronov - Pontryagin, unlike a concept of the roughness entered by the Brazilian mathematician M. Peixoto when "homeomorphism" isn't required to ε small.

Usually the roughness is considered in some closed variety or in compact area with smooth border.

As by consideration of roughness of systems it is supposed that perturbation of *DS* small in sense $C^1$, is important a concept of local roughness. Local roughness of some compact invariant set, *F* of smooth *DS* this property to keep all topological properties in some vicinity of *F* at any rather small perturbations of system.

If *F* is position of balance of "stream" (or a motionless point of "cascade", i.e. *DS* with discrete time), then the local roughness means maintaining topological properties of system at linearization in *F* point.

Necessary and sufficient conditions of local roughness for provisions of balance or motionless points (special points) answer theorem Grobman – Hartman, known of the theory of *DS*. Similar conditions are formulated also for periodic trajectories and "hyperbolic sets" which are considered further in this chapter.

### 4.1.2. Roughness of dynamic systems in the modern theory

In the general statement the roughness or structural stability of the dynamic systems (*DS*) in the modern theory is formulated by the following definitions.

**Definition 4.1.1.** For this number $z \geq 0$ two $C^r$ mappings the f: M→M and g: N→N are called "topological interfaced" if there is such homeomorphism φ: M→N that $f = \varphi^{-1} {}_o g_o \varphi$ .

**Definition 4.1.2.** Reflection g: N→N is called "factor" (or "a topological factor") mapping f: M→M if there is such a surjective continuous mapping φ: M→N that $\varphi_0 f = g_0 \varphi$. R φ is called "semi-interface".

**Definition 4.1.3.** $C^r$ is mapping *f* is called "$C^m$ is rough ($1 \leq m \leq z$) if there is such vicinity $\mathcal{U}$ mappings *f* in $C^m$, topology that each $g \in U$ mapping is topological accompanied by *f*.

**Definition 4.1.4.** $C^r$ mapping is called " $C^m$ is strongly rough if it is structurally stable and besides for any mapping $g \in U$ can be to choose the interfacing homeomorphism $\varphi = \varphi_g$ that, what in the way that $\varphi_g \varphi_g^{-1}$ evenly meet to identical mapping at approach of $g < f \ in \ C^m$ topology.

**Definition 4.1.5.** $C^r$ is the diffeomorfizm is called "topological stable" (or "rough" if it is a factor of any homeomorphism rather close to him in uniform $C^o$ topology.

Concerning streams takes place the following definitions.

**Definition 4.1.6.** $C^r$ is a stream $\varphi^t$ is called " $C^m$ rough" ($1 \leq m \leq r$) or respectively " $C^m$ is strongly rough" if any stream rather close to $\varphi^t$ in $C^m$ topology, $C^o$ is trajectory equivalent to him *C* or respectively if, besides, the discussed homeomorphism can be chosen rather close to identity for small perturbations.

**Definition 4.1.7.** Stream $\psi^t : N \to N$ is called "an orbital factor" treacle $\varphi^t$ : $M \to M$ if exists the surjective continuous $f: M \to N$ reflection which transfers orbits $\varphi^t$ to orbits $\psi^t$. $C^r$ is a stream $\varphi^t$ is called "topological stable" ("rough") if it is the orbital factor of any continuous stream rather close to him in uniform topology.

In all above formulated definitions the compactness of the corresponding phase spaces is insignificant. Besides, these definitions are fair also for cases when for some points the dynamic system is defined only on a final interval of time, such as, in the neighborhood of hyperbolic motionless a point of linear reflection. Such address leads to concepts of local roughness as it is given above.

For two-dimensional a Torus the statement is fair.

**Statement 4.1.1.** Any hyperbolic linear automorphism two-dimensional torus $C^1$ is strongly rough. A similar statement also for any *m*– measured ($m \geq 2$) the torus is right.

Important concept for determination of properties of roughness of *DS* is the concept of "a hyperbolic point".

**Definition 4.1.8.** The point of *p* is called "a hyperbolic periodic point" of *f* diffeomorfizm if $(Df^n)p : T_pM \to T_pM)$ is hyperbolic linear mapping. Her orbit is called "a hyperbolic periodic orbit".

**Definition 4.1.9.** The linear $R^n$ mapping is called "hyperbolic" if absolute values of all its eigenvalues are other than unit.

For *DS* with continuous time. It is supposed that the smooth vector field is defined in $\mathcal{U}$ and that the point orbit $\rho \in \mathcal{U}$ is in $\mathcal{U}$.

**Definition 4.1.10.** At $\xi(\rho)=0$ is point of *p* is called "a hyperbolic motionless point" (local) stream $\varphi^t$, generated by the vector field $\xi$ , if $(D\varphi_t)_p : T_pM \to T_pM$ is hyperbolic linear map for each $t \neq 0$.

At $\xi(\rho) \neq 0$ point of p is called "a hyperbolic periodic point" of *t* period for a stream $\varphi^t$ if $\varphi^t(\rho) = \rho$ and the linear operator $(D\varphi_t)_p: T_pM \to T_pM$ has unit as simple eigenvalue and at the same time has no other eigenvalues on the module equal to unit.

For the analysis of local roughness theorem Hartman - Grobman about topological associativity of mapping of the linear part near hyperbolic motionless points is very important.

**Statement 4.1.2.** (theorem Hartman – Grobman). Let sets of $U \subset R^n$ is continuously and no differentiate, also $O \in \mathcal{U}$ is a hyperbolic motionless point of $f$. Then there are such vicinities of $\mathcal{U}_1$, $\mathcal{U}_2, V_1, V_2$ point $O$ and such homeomorphism $\varphi$: $\mathcal{U}_1 \cup \mathcal{U}_2 \to V_1 \cup V_2$ that $f = \varphi^{-1} Df_o \circ \varphi$ on $U_1$, i.e. the following chart is commutative:

$$\begin{array}{ccc} f: \mathcal{U}_1 & \to & \mathcal{U}_2 \\ \varphi \downarrow & & \downarrow \varphi \\ Df_o V_1 & \to & V_2 \end{array}$$

The local roughness is defined by the following a statement and the investigation from him.

**Statement 4.1.3.** Two reversible linear squeezing mappings are typologically accompanied by identical orientation.

**Consequence 4.1.1.** Let reflection $f: U \to R^n$, $g: V \to R^n$, have hyperbolic motionless points of $p \epsilon U$ and $q \epsilon V$ respectively,

$$dim\, E^+\left(Df_p\right) = dim\ E^+\left(Dg_q\right), dim\, E^-\left(Df_p\right) = \ dim\, E^-\left(Dg_q\right), \qquad (4.1.1)$$

$$\sin n \det Df_p \middle| E^-\left(Df_p\right) = \sin \text{ndet}\, Dg_q \middle| E^-\left(Dg_q\right). \qquad (4.1.2)$$

Then there are such vicinities of $U_1 \subset U_2$ and $V_1 \subset V_2$ and such homeomorphism $\varphi$: $\mathcal{U}_1 \to V_1$, what $\varphi \circ f = g \circ \varphi$.

In (4.1.1.), (4.1.2.) space of $E+$ (.) and $E^-$ (.) are defined as the direct sums zero - spaces $E_\lambda$ of the corresponding eigenvalues $\lambda$(.), i.e. spaces of vectors $\vartheta \epsilon R^n$, satisfying to a ratio

$$((\cdot) - \lambda I)^k \vartheta = 0, \qquad (4.1.3)$$

where $k$ is some whole and also zero spaces $E_{\lambda}$, $\bar{\lambda}$ in case of in a complex interfaced eigenvalues $\bar{\lambda}$, λ:

$$E^{-}(\cdot) = \oplus E_{\lambda} \oplus E_{\lambda,\bar{\lambda}}, \quad (4.1.4)$$

$$E+(\cdot) = \oplus\ E_{\lambda} \oplus\oplus E_{\lambda,\bar{\lambda}}. \quad (4.1.5)$$

Thus, any $C^1$ is a diffeomorfizm locally rough near of any motionless point (a special point) in only case when, when a motionless point hyperbolic.

Concerning roughness of "horseshoe" transformations fairly following.

**Statement 4.1.4.** Let $\Lambda = \Pi\ f^{n}(\Delta')$ is n∊z the maximum diffeomorfizm $C^1$ is invariant relatively $f$: $N \rightarrow M$ horseshoe subset. Then for any mapping $f''$, rather close to $f$ in topology $C^1$, there is such invariant set $\Lambda$ and such homeomorphism φ: $\Lambda \rightarrow \Lambda$, что $\varphi^{\circ} f \mid \Lambda \circ \varphi$.

About roughness of hyperbolic sets fairly.

**Statement 4.1.5.** (Strong roughness of hyperbolic properties). Let $\Lambda \subset M$ is a hyperbolic set of a diffeomorfizm of $f$: $N \rightarrow M$.

Then for any open vicinity $V \subset N$ sets Λ and any δ > 0 there is it ε > 0 that if $f'$: $N \rightarrow M$ and $d_c^1(f|_v, f') < \varepsilon$, will be a hyperbolic set $\Lambda' = f'(\Lambda') \subset V$ diffeomorfizm $f'$ and such homeomorphism φ: $\Lambda' \rightarrow \Lambda$, $d_c^o(I, \varphi^{-1}) < \delta$, what $\varphi \circ f'|_{\Lambda'} = f|_{\Lambda} \circ \varphi$. Such homeomorphism φ united if δ it is enough small.

**Consequence 4.1.2.** Diffeomorfizma Anosov we are rough. Interface only if it is rather close to identical mapping.

For streams also truly following.

**Statement 4.1.6.** Let $\Lambda \subset M$ - a hyperbolic set of a smooth stream $\varphi^t$ on $M$. Then for any open vicinity ***V*** sets ***Λ*** and everyone δ > 0 exists also ε >0 that if $\Psi^t$ is other smooth stream and $d_c^1(\varphi^t, \Psi^t) < \varepsilon$, then the invariant set $\Lambda'$ for Ψ and homeomorphism φ exists: $\Lambda \rightarrow \Lambda'$, where $d_c^o(I, \varphi) + d_c^o(I, \varphi^{-1}) < \delta$, which is smooth along orbits $\varphi^t$ and sets orbital equivalence of streams $\varphi^t$, $\Psi^t$. Besides the vector field $\varphi_* \dot{\varphi}^t c^o$, is close to Ψ and if $\varphi_1$, $\varphi_2$ are two such homeomorphisms, then $\varphi_2^{-1} \circ \varphi_1$ is replacement of time of a stream $\varphi^t$ (close to identical).

**Consequence 4.1.3.** Anosov's flows are strongly rough $C^1$.

**Definition 4.1.11.** $C^1$ is a diffeomorfizm of $f$:$M$→$M$ of compact variety of $M$ is called "Anosov's diffeomorfizm" if $M$ is a hyperbolic set for $f$.

**Definition 4.1.12.** $C^1$ is stream $\varphi^t$: $M$→$M$ on compact variety of $M$ is called "Anosov's flow" if $M$ is a hyperbolic set of a stream $\varphi^t$.

Let $M$ is smooth variety, $N \subset M$ an open subset, $f$: $N \to M$ $C^1$ is diffeomorfizm on the image, $\Lambda \subset N$ is some compact, $f$ is invariant set.

Then the set $\Lambda$ is called "a hyperbolic set" of mapping $f$ if in the open vicinity $N$ sets such numbers λ,м, λ<1< м, that for any point $x \in \Lambda$ the sequence of differentials $(Df)f_x^n : Tf_x^n M \to Tf_x^{n+1} M,\ n \in Z,$ allows (λ,м) decomposition.

**Definition 4.1.14.** Let λ< м. Sequence of the reversible linear mappings $L_m$: $R^n \to R^n$, $m \in Z$, $R^n \to R^n$, $m \in Z$, allows "(λ,м) decomposition" if there are such decomposition of $R^n = E_m^+ \oplus E_m^-$, that $L_m E_m^+ = E_{m+1}^+$ and

$$||L_m|E_m^-|| \le \lambda, \quad ||L_m^{-1}|E_{m+1}^+|| \le \mu^{-1} . \qquad (4.1.6.)$$

Thus, in this section the known theoretical provisions of the modern theory of roughness of dynamic systems which in essence define only qualitative conditions of roughness of DS are considered, without considering any quantitative characteristics of roughness.

Such quantitative characteristics of roughness are very important especially at applied is more whole when it is necessary to compare systems on roughness topological or any properties of systems.

Such quantitative approach to consideration of property of roughness is offered on the basis of researches in the following section.

From the theory it is known that rough *DS* at small dimensions make everywhere sets in space of all *DS* (so-called "System Morse - Smale") supplied with $C^1$ topology are dense. At different dimensions of spaces of *DS* though there are no absolute analogs of "the systems of Morse - Smale", but some are given necessary and sufficient qualitative above a condition of roughness of *DS* which also approve density of sets of rough *DS* in a certain measure. Told above demands problem definition of comparison of rough systems on degree of roughness i.e. introduction of quantitative measures of roughness of *DS.*

### 4.1.2. Method of measures of topological roughness of DS

*A. Measure of topological roughness of DS for hyperbolic diffeomorfizm.*

Many fundamental results in the theory of roughness of *DS* have been received by A.A. Andronov and his school. In A.A. Andronov and L.S. Pontryagin's known work published in 1937 year in reports of Academy of Sciences of the USSR the concept of roughness which is called a concept of roughness according to subsequently Andronov - Pontryagin has been introduced for the first time. But this concept defines a qualitative picture of roughness, i.e. defines only ε-proximity of the homeomorphism which is carrying out equivalence of the perturbation and no perturbation systems to identical.

We will stop on a concept of roughness according to Andronov-Pontryagin. Order $n$ *DS* is considered

$$\dot{x} = F(x), \qquad (4.1.7)$$

where $x=x(t)\in R^n$ is a vector of phase coordinates, a $F$ *is* $n$-dimensional nonlinear differentiable vector function.

The system is called rough according to Andronov-Pontryagin (further rough system) in some area G if initial system (4.1.7), and the perturbation system defined in an area G subarea $\tilde{G}$

$$\dot{\tilde{x}} = F(\tilde{x}) + f(\tilde{x}), \qquad (4.1.8)$$

where $f(x)$ is a differentiable vector function, small on norm, are ε - identical, i.e. there are open $D,\tilde{D}$ areas such that $D,\tilde{D}\subset\tilde{G}\subset G$ and for them satisfy a condition: what - was ε > 0, it is possible to find it δ > 0 that if (4.1.8.) δ it is close to system (4.1.7.) in the area $\tilde{G}$, then splittings areas trajectories of systems (4.1.8) and (4.1.7.) ε are identical ("has identical topological structures and "are distorted" or "shifted" one in relation to another less, than on "ε"). This fact registers in a look

$$(\tilde{D},\ (4.1.8.))\equiv\big(D,(4.1.7)\big). \qquad (4.1.9.)$$

If the last the condition isn't satisfied, then the system will be not rough according to Andronov - Pontryagin.

We will note that determination of δ-proximity and ε-identity respectively mean: for all analytical $f_i\ (x),\ i=\overline{1,n}$ take place $\|f_i\ (x)\|< \delta$, $\|\partial f_i\ (x)/\ \partial x_j\|< \delta$, $j=\overline{1,n};$ there are biunique and continuous functions $\tilde{x}_i=\varphi_i(x)$, the systems translating each trajectory (4.1.8.) and such that

$$\| x_i-\tilde{x}_i\|< \varepsilon,$$

where $||\bullet||$ is vector norms of any look.

Thus, in this determination of roughness it is supposed that perturbations multiplicative (parametrical) and concern the right part (4.1.7.), but it is obviously possible to expand a concept assuming that perturbations of the right part are possible also additive (external, alarm). In that case, system (4.1.7.) we present in the form

$$\dot{x} = F(x, q, g), \tag{4.1.10}$$

where $q\epsilon R^p$ is a vector of the varied parameters, $g\epsilon R^l$ is a vector of external additive entrances (perturbations) of system. Then the initial system takes a form

$$\dot{x} = F(x, q_o g_o), \tag{4.1.11}$$

where $q_o\epsilon R^p$, $g_o\epsilon R^\ell$ are nominal (no perturbation) rates of vectors of $q$, $g$.

Further, in this chapter roughness of system (4.1.7.) it is understood in the sense entered above, i.e. roughness according to Andronov-Pontryagin.

### 4.2. Some concepts and definitions of the theory of dynamic systems

Here in this section we will consider some concepts and definitions of the theory of dynamic systems necessary for statement and understanding of the theory of the roughness stated in this chapter.

In the beginning without losing community (for descriptive reasons) we will consider the *DS* systems of the second order, i.e. with geometry on the plane or a look

$$\dot{x} = F_1(x, y), \quad \dot{y} = F_2(x, y), \tag{4.2.1}$$

where $x,y\epsilon R$ are phase coordinates, $F_1$, $F_2$ are continuous functions.

Then, the plane $x$, $y$ is called the *phase plane*, and each point on her represents a condition of system. The trajectory of each point ("representing" points) in the course of change of a condition of system in time carries the name of a *phase trajectory*.

If solution of system (4.2.1.) corresponding to this trajectory of $T$, it is defined for all values $t$, $-\infty<t<+\infty$, this trajectory is called the *whole* trajectory.

Sets of points $T$ at $t \geq to$ (or $t \leq to$) positive (negative) *semi-trajectory* allocated from $T$ and it designated through $T+$, ($T-$).

*Integrated curve* on phase plane which solution differential equation received by division second (4.2.1) first

$$\frac{dy}{dx} = \frac{F_2(x,y)}{F_1(x,y)}, \quad F_1(x,y) >\neq 0. \tag{4.2.2}$$

At the same time in that specific case, integrated curves and phase trajectories can coincide, and generally not.

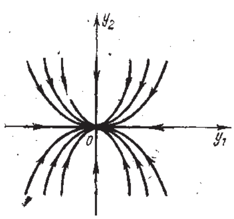
To each point ($x$, $y$) systems (4.2.1.) it is possible to compare a vector with the $F_1(x, y)$ and $F_2(x, y)$. In that case *DS* defines the *vector field* on the phase plane. Is called *special points* of the vector field or system (4.2.1) points in which $\dot{x}=\dot{y}=0$, i.e. the right part (4.2.2.) it is equal to zero and the direction of a vector vaguely

$$F_1(x,y) = 0, \quad F_2(x,y) = 0. \tag{4.2.3}$$

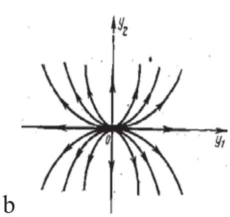
On the phase plane distinguish 6 types of special points: "knot", "dicritical knot", "degenerate knot", "saddle", "focus" and "center". These types of special points are determined by eigenvalues $\lambda_1$, $\lambda_2$ and eigenvectors $x_1$, $x_2$ matrixes of a linear part in these points

$$\mathrm{A} \triangleq \begin{vmatrix} \partial F_1/\partial x & \partial F_1/\partial y \\ \partial F_2/\partial x & \partial F_2/\partial y \end{vmatrix}_{x_0 y_0} . \tag{4.2.4}$$

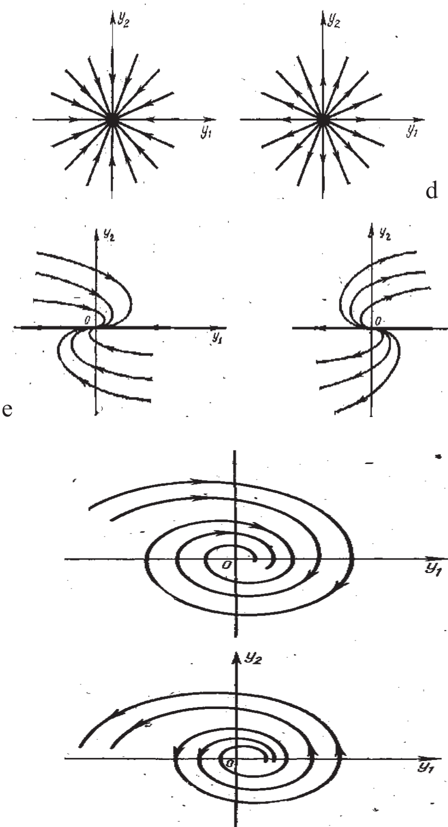
The corresponding sets of phase trajectories which are called phase portraits are shown in Fig. 4.1.




a




b




c d

e f

g h

Fig. 4.1.

At the same time for the special points (*SP*), except type "center" and "saddle" depending on the sign of the valid part $\lambda_1$, $\lambda_2$ distinguish steady and unstable *SP* (stability across Lyapunov is considered).

The special point ($x_0$, $y_0$) is called isolated if exist the vicinity of a point ($x_0$, $y_0$) in which, except this point, more any special point doesn't lie. In case all points of a curve are special points (conditions of balance) i.e. for all points is carried out (4.2.4), then such curve is called the *special line* of system.

One of ways of creation of phase portraits is the method an *isoclinal. Isoclinals* are called the curves determined by ratios:

$$F_1(x,y) + c_1F_2(x,y) = 0, \quad F_2(x,y) + c_2F_1 = 0, \tag{4.2.5}$$

where $c_1$, $c_2$ are constants in which all points the directions of tangents to trajectories are identical. At $c_1 = 0$ we receive to an isoclinal of vertical inclinations, and at $c_2 = 0$ horizontal inclinations.

In case of linear *DS* the type of a special point defines the movement of system at any deviations from a special point. For nonlinear systems the type of a special point defines behavior of phase trajectories only in some small vicinity *SP*.

At a research of nonlinear systems an important role is played by special trajectories. Treat them: except special points these are the isolated closed trajectories which are called *limit cycles* and special lines are *separatrixes.*

Limit cycles can be both stable, and unstable. Stable limit cycles have received the name of *self-oscillations*. When speak about stability of limit cycles, in old sense speak only about *orbital stability*.

The limit cycle is called orbital stable if as for a positive semi-trajectory of *T*+, and the negative semi-trajectory of *T* - at any set $\varepsilon > 0$ can specify $\delta > 0$, it that at any trajectory of $T'$ passing at $t=t_0$ through any point of $M'$ the vicinity of $M$ belonging $\delta'$ the $T'^{+}$ semi-trajectory (according to $T'^{-}$). Any trajectory which isn't orbital stable is *orbital - unstable* or *special*. We will note that orbital stability differs from stability across Lyapunov.

Separatrixes divide the phase plane into areas with phase trajectories of various types. In the neighborhood of a special point like "saddle" of a separatrix is asymptotes and are called also *moustaches of saddles.* Special points divide area into subareas which points are points of nonspecial trajectories. Such areas are called *elementary cells* (or just *cells*) dynamic system on the phase plane.

Important concepts of a qualitative research of dynamic systems are concepts of homeomorphism and identity.

**Definition 4.2.1.** *Topological mapping* or *homeomorphism* of the plane (area) in itself is called biunique and bilaterally continuous mapping of the plane (area), and geometrical images which can be received the friend from the friend by topological mapping is *gomeomorfly.*

At all possible topological mappings some lines of splitting the phase plane into trajectories can will change, and others can remain invariable or *topological invariant*: for example, the closed trajectory remains closed, there is a number and types of special points, there are invariable types of cells, etc. For the comparative characteristic of topological structures of *DS* there is a concept of identity.

**Definition 4.2.2.** Two topological structures, or otherwise qualitative pictures of splitting the phase plane into the trajectories (or areas on a trajectory) set by two systems of a look (4.1.12.) are called *identical* if exist homeomorphism at which trajectories of one system are mappinged in the same trajectories another. At the same time the set of data on the nature of equilibrium state (about type of special points) a relative positioning of limit cycles and the course of separatrixes is called the *scheme of breaking on a trajectory*.

In case of the systems of a high order ($n \geq 3$) in phase space, the character (type) of special points, special lines and limit cycles becomes complicated and it isn't always as simple to speak about concrete types of points, as on the phase plane, at least because at the same time eigenvalues of a matrix linear it is frequent on two, and a set and which of them definitely characterizes type of a special point.

At $n \geq 3$ in dynamic systems there are phenomena impossible for two-dimensional systems. Emergence of *chaotic oscillations* and *strange attractors* (the attracting varieties) in them which appear in phase spaces with unstable behavior of trajectories belong to such phenomena. These phenomena can arise at changes of parameters of systems of the *nonlinear* differential equations, in particular, their emergence is connected with emergence of turbulence.

The term "strange attractor" has been entered by D.Ruelle and F.Takens in 1970, but before the works connected with model researches E. Lorenz it was a little used in scientific literature. At the same time the term "attractor" means any attracting variety (a continuous set), for example, equilibrium state (special points or *SP*), limit cycles, and the term "strange attractor" means the quasiperiodic movements to them and has taken roots only after emergence of interest in work of the American meteorologist E. Lorenz of 1963 where was considered model of dynamics of the atmosphere in which at certain parameters in some limited area of phase space there are chaotic oscillations called now by "a strange attractor of Lorenz" (sometimes simply "Lorenz's attractor").

The theory of dynamic systems is most developed, in particular, the theory of roughness of *DS*, only for $n = 2$ case on the phase plane, for high orders ($n \geq 3$) active researches, first of all by mathematicians, especially in new promising scientific and applied aspects the directions connected with studying of strange attractors, bifurcations and accidents are conducted now.

At the same time, on the review of literature which is carried out by the author in the field of the theory of dynamic systems it is possible to claim that effective methods of a quantitative research of *DS* properties, in particular the property of roughness, doesn't exist now. But there is a search and development of such methods. One of attempts of completion of such gap in the theory of roughness of *DS* is also made on the basis of the theory offered by the author in this work which is called "the theory of conditionality of topological roughness" or in brief "the theory of the topological roughness" which is based on the corresponding method of "topological roughness".

**4.3. Bifurcation of dynamic systems**

The concept of bifurcation of these systems also is closely connected with a concept of roughness of dynamic systems. As already earlier it was noted, the term bifurcation means "bifurcation" and belongs to any spasmodic change happening at the main change of parameters in any system, whether it be dynamic, ecological, economic, synergetic, etc.

The beginning of works on the theory of bifurcations should be carried to H. Poincare's works where he investigated dependences of conditions of balance on parameter. The significant contribution to the theory of bifurcations was made also by the American scientist E. Hopf.

The huge contribution to the theory of bifurcations was made by A.A Andronov and his school. In essence they have considered all questions of bifurcations on the phase plane: bifurcations of provisions of balance (special points), bifurcations of limit cycles, etc.

Much attention is paid to questions of bifurcations in works of V.I. Arnold, D.V. Anosov and their colleagues. In these works researches of bifurcations and features for big orders of systems ($n \geq 3$), on the basis of modern topological methods are already conducted.

### 4.3.1. Basic concepts and definitions of the theory of bifurcation of dynamic systems

At expanded and strict definition of a concept of bifurcation the following takes place.

**Definition 4.3.1.** Values of parameter $q=q_o$ (both scalar, and vector) on which depends some qualitative property $S=S(q)$ is called ordinary if exists final small $\varepsilon > 0$, it that for all $q$ satisfying to inequality

$$\| q - q_o \| < \varepsilon, \qquad (4.3.1.)$$

the similarity condition is satisfied

$$S(q) \equiv S(q_0). \qquad (4.3.2.)$$

If there are no vicinities $q = q_0$ for which it is carried out (4.3.2.), that such value of parameter $q$ is called *bifurcation,* and the phenomenon corresponding to this value of parameter $q$ is called *bifurcation*.

In case of dynamic systems in phase space as the considered property accept topological structure of splitting phase space into integrated curves. In many sources of foreign literature instead of the term bifurcation often meets the term *catastrophe*, in particular, in works on the theory of features of reflections.

The value of the theory of bifurcations for a research of dynamic systems is immutable that it is obvious to cite, for example, enough known paraphrased H. Poincare's words about that, "that bifurcations as torches, light a way from the studied dynamic systems to unexplored".

On the phase plane can be bifurcations: emergence or disappearance and special points (conditions of balance), limit cycles, merge or division of separatrixes, etc. At the same time bifurcation of emergence or disappearance of limit cycles carries the special name of bifurcation of Poincare – Andronov - Hopf on names of the scientific bifurcations which have made the greatest contribution to the theory and investigating this type of bifurcation historically the first and is the fullest. But in modern literature the name of this bifurcation as bifurcations of Hopf to whom we will adhere in this work has most become stronger.

In multidimensional dynamic systems also more difficult bifurcations, in particular, bifurcations of emergence or disappearance of various invariant sets, for example, of so-called *strange attractors* are possible.

The attracting varieties in phase space in which the movements of phase trajectories have chaotic character are called *strange attractors*, i.e. the investigating coordinates of points of trajectories aren't determined by the previous coordinates of points (unpredictability of behavior of trajectories). In the last decades attach to studying of such phenomena connected with chaos and turbulentness huge significance, as from the informative point of view, and the applied point of view. It first of all is connected with the generalizing world outlook role of knowledge of the phenomena of chaos and an order.

For studying of bifurcations various methods of generally qualitative research of systems, for example, methods of the theory of features of continuous mappings or in the simplest case the theory of dependences of equilibrium state on parameter. Quantitative analogical methods of researches it is used seldom and as a rule for the systems of a low order.

Distinguish *local* and *not local bifurcations*.

Local bifurcations it is such bifurcations of phase portraits which happen near special points and limit cycles, and not local respectively far from the last.

We will provide some terms and definitions from the theory of bifurcations.

**Definitions 4.3.2.** *A hyperbolic special point* is called the special point in which any of eigenvalues of a linear part of system doesn't lie on an imaginary axis according to *not hyperbolic the special point* in which any eigenvalue lies on an imaginary axis is called.

In one-parametrical sets of vector fields not hyperbolic special points of two types meet: one eigenvalue of a special point is equal to zero or two is clean imaginary other than zero eigenvalues, and the others don't lie on an imaginary axis.

**Definition 4.3.3**. *Bifurcation diagram* is called the diagram in which insertion of bifurcation solutions of the equations of dynamic systems depending on parameters with the indication of their stability and instability is shown.

**Definition 4.3.4.** *Bifurcation equation* is called the equation which determines "amplitudes" of branchings of solutions of the equations of dynamic systems near a bifurcation point.

For an explanation of sense of definitions 4.3.3 and 4.3.4 we will give widely known example.

Let the system be described by the equation

$$\dot{x} = F(x, q), \tag{4.3.3}$$

where $q$ is the parameter, $x \in R^n$ *is* a vector of states.

Further let for $i$ that coordinate of a condition of system (4.3.3) we have the equation

$$\dot{x}_i = F_i(x, q). \tag{4.3.4}$$

Then in the neighborhood of some bifurcation value of parameter $q = q_c$ we will have decisions

$$x_i\,(t, q) = x_{oi}(q_c) + a_i \varphi_i(t), \tag{4.3.5}$$

where $x_{oi}(q_c)$ is a steady (equilibrium) state at $q = q_c$, $a_i$ is "amplitudes", detection of decisions $x_i\,(t, q)$, $\varphi_i(t)$ are functions of branchings (4.3.3) near a bifurcation point $q = q_c$. At the same time $a_i$ determined "amplitudes" from the bifurcation equations

$$\dot{a}_i = G(a_i, \rho)\ . \tag{4.3.6}$$

For an example of the equation

$$\dot{a} = -a^3 + qa, \tag{4.3.7}$$

we will have two stationary decisions

$$a=0 \text{ и } a = \pm\sqrt{q}, \qquad (4.3.8)$$

and the bifurcation diagram is submitted in Fig. 4.2.

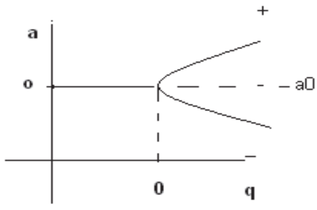



Fig. 4.2.

## 4.4. Concept of "typicality" of systems

*Ratio of "roughness", "typicality" and "giperbolichness".*

*"Typicality"* is one of fundamental concepts of modern mathematics. *"Typicality"* is defined how the property of systems belongs to some stable continuous set of systems. Sometimes about "typicality" speak as about property of systems of "general provision", i.e. about property of characteristic systems for some set of mathematical objects.

For example, rough systems on the plane form "typical" systems and are the dense set opened everywhere in space of all systems of the considered type, the supplied $C^1$ topology. Andronov A.A. school it is proved coincidence of concepts of roughness and "typicality" for the systems of a low order $n \leq 2$ completely coincide. For the systems of higher order $n \geq 3$ these concepts not always coincide and have the differences. From the point of view of "typicality" *bifurcation* is defined as "reorganization" of the typical object depending on the parameter (parameters) at critical values of this parameter.

Using a concept of "typicality" it is possible to speak also about "typical" bifurcations of systems. Actively used the idea of "typicality" in the works and the French scientist R. Thom for a conclusion of results on the theory of catastroph, at the same time in essence describing sets of "typical" bifurcations for the description of so-called elementary catastrophs. The last in effect bifurcations of "typical" families of the systems depending on a large number of parameters.

Proceeding from "typicality" on the plane there are only five "typical" bifurcations. Three of them are local bifurcations it is the birth of degenerate position of the balance (a special point) which is breaking up on two hyperbolic ("saddle" and "knot"); the birth of a limit cycle from degenerate focus; the birth of the degenerate limit cycle which is breaking up on two hyperbolic ("focus", "saddle"). Two other bifurcations of global character this birth of a limit cycle from seperatrix loop a saddle or from a separatrix loop a saddle - knot and they treat degenerate situations.

In a multidimensional case, a trajectory of systems become more difficult. In this case the "difficult" behavior of trajectories isn't a rare phenomenon, and "simple" behavior characterize only this way the called systems of Morse - Smale.

**Definition 4.4.1.** *The system (a stream, the cascade) of Morse - Smale* is called the smooth dynamic system (a stream or the cascade) set on the closed variety of $M$ if all her trajectories aspire (in both parties in time) to some periodic trajectories (including provisions of balance in case of a stream treacle and motionless points in case of the cascade), and periodic trajectories final number and all of them hyperbolic, and their invariant steady and unstable varieties are in the general provision, i.e. are crossed only is transversal. At the same time a diffeomorfizm μ at which iteration procedure the Morse - Smale cascade turns out is called the *diffeomorfizm Morse - Smale.*

Systems such have been introduced by S. Smale in 1960. He has received inequalities connecting number of periodic trajectories of various types with gomologiya of phase variety. The known inequalities of P.Morse for number of critical points of smooth function $f$ can be considered as the special case which is turning out when the dynamic system is a stream of $\dot{x} = \text{grad}\, f$.

The systems of Morse - Smale represent in essence natural generalization on a multidimensional case of conditions which have been specified by A.Andronov and L.Pontryagin as conditions of roughness of streams the planes. But as the first substantial result S.Smale reminded results of P.Morse of the systems of Morse and wasn't related to roughness, the name of systems of Morse and Smale has taken roots.

At the same time S.Smale, from the very beginning meant communication with the theory of rough systems and assumed that the systems of Morse – Smale are rough. It was later is proved by S.Smale and J. Palis.

In the systems of Morse – Smale the importance knows about "not wandering trajectories"

**Definition 4.4.2.** The point $\partial$ is called "*not wandering*" if for any her vicinity of $A$ and any T > 0 there is t > T that $\varphi_t A \cap A \neq \emptyset$. Then all points $\varphi_t^{\partial}$ is "not wandering"; the corresponding trajectory $\{\varphi_t^{\partial}\}$ is called "not wandering trajectory".

Not wandering points form the closed invariant subset Ω phase spaces.

In the system of Morse – Smale only periodic trajectories are not wandering (including provisions of balance). Therefore the system of Morse – Smale is defined still as system at which the set of not wandering points consists of final number hyperbolic periodical trajectories which invariant varieties are crossed only it is transversal.

In case of small dimension of phase space rough systems are in the accuracy of system of Morse – Smale. But in multidimensional couples there are rough systems, more difficult character. Such case has been found by Smale, on the example of so-called "Smale's horseshoe", where the rough system with infinite number of periodic trajectories is observed. Such "difficult" rough systems differ from the systems of Morse – Smale generally in the fact that that role which is played in them by periodic trajectories in "difficult" rough systems, is played by so-called hyperbolic sets.

For the smooth cascade $\{f^n\}$ with the closed phase variety of $G$ the following takes place.

**Definition 4.4.3.** Compact invariant (consisting of the whole trajectories) the set of *M∈G* is called hyperbolic if for each point of *x∈M* the tangent space of $T_xG$ decays in the direct sum

$$G = E_x^s \oplus E_x^u \,, \qquad (4.4.1)$$

in such a way that at $\xi \in E_x^s, \quad \eta \in E_x^u, \quad n \geq 0$ :

а) $|(f^n)'\xi| \leq a|\xi|e^{-cn}, \quad |(f^{-n})'\xi| \geq b|\xi|e^{cn};$

б) $|(f^n)'\eta| \geq b|\eta|e^{cn}, \quad |(f^{-n})'\eta| \leq a|\eta|e^{-cn}$ ,

where *a, b, c* are the positive constants, which aren't depending from *x, η, ξ, n*. Subspace $E_x^s$ *is called stable (contracting), and* $E_x^u$ *is unstable (extending);*

These subspaces are defined by the properties a), b) is unambiguous, their dimensions are locally constant, and they depend from *x* is continuous.

For a smooth stream $\dot{x} = F(x)$ on *G.*

**Definition 4.4.4.** The compact invariant set of *M⊂G* is called hyperbolic if all belonging to it balance position (when those are available) are hyperbolical (their final number), a set

$$B = M \setminus \{x = f(x) = 0\},$$

is closed and for each point of *x∈B* the tangent space of $T_x\,G$ decays in the direct sum

$$T_xG = E_x^s \oplus E_x^u \oplus Rf(x), \qquad (4.4.2)$$

in such a way that at $\xi \in E_x^s, \quad \eta \in E_x^u, \quad t \geq 0$ :

a) $|\varphi_t'\xi| \leq a|\xi|e^{-ct}, \quad |\varphi_{-t}'\xi| \leq в|\xi|e^{-ct};$

b) $|\varphi_t'\eta| \leq в|\eta|e^{ct}, \quad |\varphi_{-t}'\eta| \leq a|\eta|e^{-ct}$ ,

where *a, b, c* are the positive constants, which aren't depending from *x, η, ξ, t.*

Generally, a final set of periodic trajectories (to which also hyperbolic provisions of balance in case of a stream belong) form a hyperbolic set. The private case, when all phase variety is a hyperbolic set.

Systems with the last case have been introduced and studied by the famous mathematician D.V. Anosov which are called by him *U-systems*, but in literature carry also other name of *systems (streams, cascades, diffeomorfizm) of Anosov.*

Generally definition of a hyperbolic set has been given by S. Smale. It is proved that, Anosov's systems rough.

In the hyperbolic theory the conditions of roughness of dynamic systems determined by Smale's hypothesis are known. According to this hypothesis the roughness of dynamic systems requires also enough that the set of not wandering points was hyperbolic and periodic trajectories were dense in him, and stable and unstable varieties of not wandering trajectories were crossed everywhere is transversal. Smale has proved that when performing these conditions a structure of a set of not wandering points, and then and all qualitative picture it is possible to investigate rather in detail.

The sufficiency of conditions of roughness is proved by R.Robinson. As for conditions of roughness are proved not for the general case, and in special cases of systems of small dimension.

The question of need for a multidimensional case remains open, owing to the fact that as Smale has shown it is connected with a question of "typical" properties of dynamic systems which aren't completely disclosed. Smale has so shown that in a multidimensional case rough systems aren't dense in space of all systems.

So far it wasn't succeeded to find yet such conditions for multidimensional systems which like roughness in case of small dimension would be satisfied for "typical" system and would define its possible properties (an example Lorenz's attractor).

Thus, a question of qualitative property of roughness, first of all it is connected with a question of "typicality" though was mentioned in a multidimensional case as a little earlier differs from a case of systems of small dimension, namely such systems form the dense set which isn't opened everywhere, and form the set containing everywhere dense local sets.

In the theory of rough systems the following two main results about "typical" properties of dynamic systems are known.

1. **Theorem of Kupka - Smale:** In space of all smooth dynamic the systems

(streams or cascades) of a class $C^z$, z≥1, on some phase variety "are typical systems" which have all periodic trajectories (including provisions of balance (special points)) are hyperbolic, and invariant steady and unstable varieties of these trajectories are crossed everywhere is transversal. Systems with such properties are called the *systems (streams, cascades, diffeomorfizm) of Kupka-Smale*, i.e. Kupka-Smale's systems "are typical". Follows from this theorem that the rough system has to be Kupka-Smale's system.

**2. Lemma about short circuit (Lemm of Ch. Pyyu):** if *a* is not wandering point of smooth dynamic system, then as much as close in sense of $C^1$ to this system exists system for which *a* is periodic.

The **consequence** from this lemma: in space of everything, the dynamic systems of the class $C^1$ systems which have periodic points (including provisions of balance (special points)) "are typical" everywhere are dense in a set of not wandering points. In case of $C^2$ this lemma isn't proved and fair only for $C^1$.
If addresses physical roots of a concept of roughness of systems, then as is well-known from Andronov and Pontryagin's work such systems are associated systems with "simple" behavior of trajectories.

In connection with told above many researchers ask a question and that has given the generalizing mathematical formulation of the question about roughness when are considered "difficult" behavior of trajectories. On this question believe that new and interesting it is a little. So, it is very difficult to check sufficient conditions of roughness in concrete cases. In particular the hyperbolicness of all set of not wandering points difficult is established. It is even more difficult to establish transversality of invariant steady and unstable varieties. But of course, the local roughness of hyperbolic sets is quite easily found, and such examples are enough in modern literature according to the theory of roughness and his applications.

At the same time, development of new sections of the theory of roughness has led to emergence of the modern "hyperbolic" theory and also contributed to the development of the *theory of bifurcations*. And these two theories are very important applied aspect of science.

Systems with simple behavior of trajectories deserve attention though because it is essential that, systems with such character of behavior these are the systems of Morse - Smale are rough. But at the same time, also the fact that in space of dynamic systems there are areas entirely filled with systems with difficult behavior of trajectories is essential (for example, system with Lorentz's attractors). The fact that one of systems with difficult behavior of trajectories rough, and others are not rough and that as those, and others fill some areas, not so significantly.

It is known that in space of dynamic systems there are areas entirely filled with systems at which some details of a qualitative picture of behavior of trajectories change at as much as small perturbations in other words are filled with not rough systems. But in the known examples of such extreme it is sensitive to change of system some, "thin" details of a qualitative picture possess. In real physical systems these details "are washed away" from behind external and internal noise therefore in reality it is perhaps not so important whether they remain at small perturbations or not.

Told, results in the idea to modify a concept of rough system so that at rough system at small perturbations not all qualitative properties but only what - that the main, then perhaps such systems, will be "typical" remained. But such result concerning "typicality" and "roughness" isn't received.

In certain examples of not rough systems some is observed a hyperbolicness, weaker, than in hyperbolic sets, but isn't available the general formulation which would cover all these cases of a hyperbolicness.

# CHAPTER 5. CHAOS IN DYNAMIC SYSTEMS

## 5.1. Emergence of chaotic fluctuations (chaos)

The systems which aren't meeting the roughness conditions stated in paragraph 4.1. are called not rough. By numerous researches it is established that sets of not rough systems, as well as rough systems form continuous sets in spaces of dynamic systems. It is also known that sets of rough and not rough systems are divided from each other by points and the fields of bifurcation, i.e. through bifurcations of system pass not only from one area (set) of rough systems to another but also from rough to not rough systems and vice versa.

Bright example of not rough systems are systems with a balance point like "center". In terms of the theory of dynamic systems not rough systems it is systems with not hyperbolic special points.

One of the phenomena in the synergetic systems causing huge attention of researchers in various areas, sciences are the so-called strange attractors representing the attracting varieties in phase space with chaotic behavior (chaos) of trajectories in these varieties. The research of strange attractors draw interest and therefore that many scientists see in studying of this phenomenon a clue of mysteries of the nature of turbulence and chaos in the systems of various physical nature are physical, chemical, biological and also in economic and social systems. There is relevant also a task of controlling of chaos in synergetic systems.

From the point of view of the theory of roughness system with the chaotic movement (chaos) not rough system, i.e. thus, strange attractors define sets of not rough systems.

The first classical example of chaotic systems with strange attractors was the system with Lorenz's model opened 1963 by the American meteorologist Edward Lorenz (Massachusetts Institute of Technology).

E. Lorenz researching atmospheric currents for forecasting of weather had received model of the thermal convention in the atmosphere in the form of the system of the third order

$$\begin{cases} \dot{x} = 6(y - x), \\ \dot{y} = \rho x - y - xz, \\ \dot{z} = xy - \beta z, \end{cases} \qquad (5.1.1)$$

where *x* is the variable proportional to amplitude of speed of the movement, and variables *y, z* reflect distribution of temperature in a convective ring, parameter Ϭ is represents Prandtl's number, ρ is Rayleigh's number, and β = 8/3 is a geometrical multiplier.

Most often believe Ϭ = 10, and the operating parameter is quantity ρ.

It is established that the strange attractor of Lorenz with the chaotic movement in phase space, arises at value ρ = 24,74, behind a point of bifurcation of Poincare – Andronov - Hopf (Hopf) in system (5.1.1). At the same time in phase space of system there are two unstable saddle-fokus and one saddle point around which there are chaotic movements in many synergetic systems, in particular, such movements arise in gas lasers.

Chaos arises in many synergetic systems with various physical nature, but there are some standard scenarios of transition to the chaotic movements.

The following scenarios of transition to chaos are characteristic:

A. Transition to chaos via the *infinite cascade of bifurcations of doubling of the period* (the *universal scenario of M. Feigenbaum*);

B. Transition through *alternation*;

C. Transition through *Hopf's bifurcations*.

**A.** The *universal scenario* of transition through the infinite sequence of doubling of the period has been opened by the American scientist in the field of mathematical and theoretical physics *Mitchell Feigenbaum* in 1976.

By the time of Feigenbaum's opening it was known that in discrete displays of type

$$x_{n+1} = \lambda f(x_n), \qquad (5.1.2)$$

at change of parameter λ > 0 existing cycle having the *T* period loses stability, and stable is a cycle with the period *2T*, then *4T*, etc. (Fig. 5.1). The interval of change of parameter λ within which the period cycle $2^n$ is stable is quickly narrowed. For example, at function of the right part (5.1.2) in a look

$$f(x_n) = \lambda x_n(1 - x_n), \tag{5.1.3}$$

at 1< λ <3 the logical equation (5.1.3) has two special points: $x = 0$; and $x = (\lambda-1)/\lambda$, at the same time the beginning of coordinates is the unstable point, and the second point is stable.

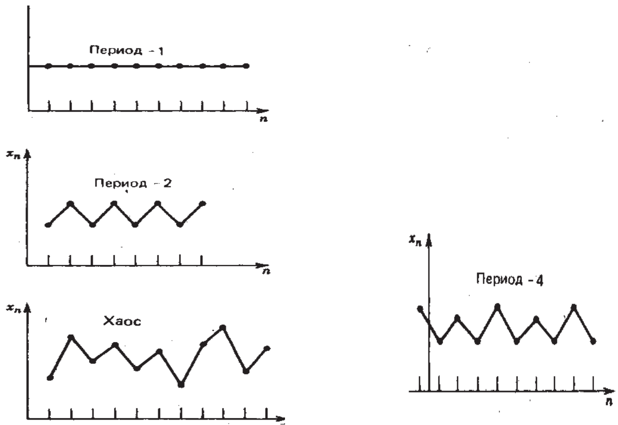



Fig. 5.1.

At λ = 3 inclination at $x = (\lambda-1)/\lambda$, exceeds unit ($f'$=2-λ) and both points of balance become unstable. At 3< λ <4 this simple differential equation describe a set of multiperiodic and chaotic movements.

Process of transition continues until λ doesn't reach value $\lambda_\infty$=3.56994 …. Near this value λ at which occur doubling of the period submits to the exact law

$$\frac{\lambda_{n+1}-\lambda_n}{\lambda_n-\lambda_{n-1}} \to 4{,}66920 \ldots \tag{5.1.4}$$

This limit relation is called Feigenbaum's number.

At values λ, $\lambda_\infty$ , there can be chaotic iterations, i.e. the behavior of model on big times doesn't keep within a framework of simple periodic movements, i.e. there is chaos.

By researches it is shown, as other displays of a look (5.1.2) where *f(x)* is square or more difficult function, behave similarly, satisfying the same law (5.1.4).

Phenomena of doubling of the period or regular change bifurcation parameter is called the universal scenario Feigenbaum's on emergence of chaos which is characteristic also to continuous systems, in particular Rösler's system.

**B.** The transition to chaos which is most often found in applications *alternation* (it is revealed the French physics P. Manneville and Y. Pomeau, 1980). The alternation looks as constant (at change of parameter) disappearance of periodic oscillations due to their interruption chaotic splashes. In process of change of parameter flash of chaos become more and more frequent and long. For example, it was investigated convection in a cell with temperature gradient (convection Rayleigh - Benard) or hydrodynamic system.

At the same time the average duration of a chaotic or turbulent phase of the movement τ definitely changes with change of some parameter of system, for example, the dependence has an appearance

$$\tau \approx \frac{1}{(\lambda-\lambda_c)^{\frac{1}{2}}}, \qquad (5.1.7)$$

where $\lambda_c$ is value at which the periodic movement becomes chaotic.

Transition to chaos through alternation in hydrodynamic system is connected with merge and the subsequent disappearance of stable and unstable periodic trajectories.

Transition to chaos through alternation is possible also in Lorenz's system at rather large numbers of Rayleigh (ρ = 166,2; б =10; β=8/3).

**C**. Transition to chaos through *bifurcation Hopfa (Poincare-Andronov-Hopfa)* has been considered on the example of the system (attractor) of Lorenz (5.1.1). This scenario of transition to chaos is also standard (characteristic) of many synergetic systems.

As a rule a harbinger of such scenario of transition to chaos is presence at the system of two or more simultaneous periodic oscillations. When frequencies of these oscillations $\omega_1$ and $\omega_2$ are incommensurable, the observed movement on a being not periodically, i.e. kvaziperiodically. Such movements can be presented occurring on a surface a torus in phase space.

In such system the chaotic movements are characterized by destruction of quasiperiodic toroidal structure at change of parameters of system.

In some synergetic systems at different values of parameters it is possible to observe all three types of scenarios emergence of chaos.

Also the return transitions from chaotic movements to periodic when chaotic oscillations, having arisen at certain values of parameters, through a certain period degenerate in the periodic or quasiperiodic movement are known to researchers of chaos. This transitional chaos a consequence of bifurcation or sudden disappearance of the established chaotic oscillation.

Chaos is possible not only in dissipative systems, but also in conservative systems. Moreover, as is well-known finding solutions to the equations of heavenly mechanics has brought in the end of the 19th century Poincare, to the assumption that solutions of many problems of dynamics are sensitive to entry conditions and motions of bodies on orbits are unpredictable. At the same time physical examples of conservative systems are connected with problems of calculation of orbits in heavenly mechanics and behavior of particles in electromagnetic fields.

Though as a rule, in real terrestrial conditions in dynamic systems there are energy losses, in some of them, for example, in the structured designs or microwave resonators, attenuation isn't enough, and on final intervals of time they can be considered conservative or Hamilton systems.

In conservative systems, the same types of limited oscillating motions, as in dissipative systems are found, i.e. the periodic, subharmonic, quasiperiodic and chaotic movements are possible. The main difference between oscillations in dissipative and conservative systems is that chaotic oscillations in dissipative systems (with energy loss) find fractal structure unlike conservative where there is no such structure.

Conservative systems are characterized by the uniform density of probabilities in limited areas of phase space and have other reflection of Poincare, than dissipative systems. But the measure of a divergence of close oscillations as Lyapunov's indicators is also applicable to them.

**5.2. Systems with chaotic oscillations**

Models of numerous examples of systems where chaotic oscillations or chaos are found have been presented in Chap. 3 by consideration of synergetic systems. These are such systems as Lorenz, Lengford, Rössler, Belousov-Zhabotinsky system, Chua’s circuit, oscillators Van der Pol and Duffing and others. From examples of discrete mapping, are characteristic the logistic equation describing growth of populations and also the Henon mapping of the horseshoes type ("Smale's horseshoe") and transformation of "baker".

Classical examples of systems with chaotic dynamics are the system or Lorenz's attractor and the logistic equation. These two examples are characterized by many features of chaotic dynamics, such as subharmonic bifurcations, the running-up trajectories, doubling of the period, Poincare's map and fractal dimensions.

In this paragraph we will consider some widely known systems with chaotic dynamics.

*Lorenz's system.*

The system or Lorenz's attractor is historically the first model of systems with chaotic dynamics where in essence in the determined system arise unpredictable it would seem casual oscillations, in some limited area of phase space called Lorenz's "attractor".

Lorenz's system has been received as a result of modeling of dynamics of the atmosphere.

The liquid layer which is under the influence of gravity and which is warmed up from below is considered. Across this layer there is some difference of temperatures. At achievement of a difference of temperatures of rather big size, arise circulating, liquid whirls in which warm air (liquid) rises, and cold falls. The two-dimensional convective current is described by means of the classical equation of Navier-Stokes.

At the small differences of temperatures ***Δ*** liquid isn't mobile *T*, but at some critical value ***Δ*** *T* arises a convective, i.e. circulating current. This movement is called *convection Rayleigh – Benard*.

Lorenz investigating Navier-Stokes equation decomposing on fourier-harmonicas along the spatial directions left three harmonicas.

As a result, in a dimensionless form Lorenz (5.1.1) equations which at a set of parameters σ = 10, β = 8/3 and varied ρ have an appearance are received

$$\begin{cases} \dot{x} = 10(y - x), \\ \dot{y} = \rho x - y - xz, \\ \dot{z} = xy - 8/3z. \end{cases} \qquad (5.2.1)$$

In system (5.2.1) at ρ < r1 = 1,0 exists the only special point of $SP_1$ (0,0,0) types "stable knot". Further at $\rho = r_1$ =1,0 in system (5.2.1) are points $SP_2$ ($[8/3(\rho - 1)]^{1/2}$, $[8/3(\rho - 1)]^{1/2}$, ρ – 1) and $SP_3$ ($-[8/3(\rho - 1)]^{1/2}$, $-[8/3(\rho - 1)]^{1/2}$, ρ – 1) like "unstable focuses" appear two more symmetric relatively. Same time, $\rho > r_1$ turns into a special point like "saddle-knot".

At value $\rho = r_f$ =1,345 in system (5.2.1) occurs bifurcation of change like special points of $SP_2$ and $SP_3$, namely, the last turn into special points like "stable focuses". At $\rho = r_2$ =13,926 in the considered system there is metastable chaos when attractors of $SP_2$ and $SP_3$ from global turn into local attractors with some areas of an attraction.

Further, at $\rho = r_3 \approx 24{,}74$, in system occurs bifurcation Poincare – Andronov – Hopf (Hopf) when eigenvalues $\lambda_{2,3}$ in special points of $SP_{2,3}$ become purely imaginary. The value of parameter $\rho = r_3$ draws great attention of researchers with the fact that at $\rho > r_3$ in system (5.2.1) arises the interesting phenomenon called "a strange attractor" of Lorenz. Here, in limited area of space $R_3$ around unstable saddle-fokus of $SP_{2,3}$ there are chaotic oscillations covering a saddle point of $SP_1$. In system there are also other points and types of bifurcations which in this work aren't considered.

It is established that in a strange attractor of Lorenz the movement is globally limited in ellipsoidal area of phase space.

*Logistic map (equation).*

It is known that the logistic equation has received the name in connection with a task about livelihood of population of animals (*logistics* is supply, livelihood) and is the simplified model of dynamics of populations.

Let $x_n$ represent number of individuals in the isolated territory in a year with number $n$, divided into the maximum number of individuals which this territory is capable to support. Population number next year is $x_{n+1}$, depends on that how many individuals were this year, i.e. from $x_n$. This dependence is represented logistic mapping of a look (5.1.3):

$$f(x_n) = \lambda x_n(1 - x_n), \quad 0 \leq x_n \leq 1. \qquad (5.2.2)$$

Obviously, size $1 - x_n$, is proportional to amount of the available food. Or otherwise, in process of approach of number of populations of $x_n$ to critical value 1, amount of food, being constantly reduced, approaches zero. The physical sense of parameter λ represents fertility of population. The more the value λ, the quicker population will recover after any catastrophs. But great values of parameter λ as a rule lead to chaotic populations. The main conclusions connected with the chaotic movement in the systems described by the logistic equation are given in the previous paragraph 5.1., where it is noted that chaos in the system of the logistic equation arises through the sequence of doubling of the period, characterized by Feigenbaum's number (5.1.4).

*Nonlinear electric circuit.*

The example of chaos in electric circuit has been shown by the Japanese scientist Ueda in a circuit with a nonlinear inductive element. This circuit is described by the following equation

$$\ddot{x} + k\dot{x} + x^3 = r \cos t, \qquad (5.2.3)$$

which in essence is a special case of the equation of Duffing. By means of modeling on analog and digital computer facilities of Ueda has received chaotic dynamics of system (5.2.3) which contours of map of Poincare are shown in Fig. 5.2.

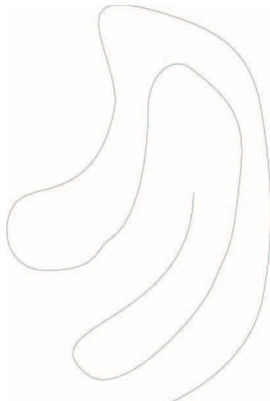


Fig. 5.2.

*Chaos in control systems.*

In control systems the chaotic movements and oscillations are also possible.

We will consider the system of the third order described by the following equations

$$m\ddot{x} + \delta\dot{x} + f(x) = -y, \quad \dot{y} + \alpha y = k_1[x - x_g(t)] + k_2\dot{x}, \qquad (5.2.4)$$

where $y$ is the size of force created by a feedback loop, and $k_1$ and $k_2$ respectively feedback coefficients by situation and speed $x_g(t)$ is an external basic signal.

For system (5.2.4) two types of tasks for researches are possible. In the first case, believe that the system is autonomous, i.e. a basic signal of zero $x_g(t) = 0$. In this case, the space of coefficients of $k_1$ and $k_2$ for search of areas of balance, periodic or chaotic oscillations is investigated. In the second case, a signal of $x_g(t)$ periodic, i.e. weight moves on the set trajectory periodically. Then, values of frequency and coefficient of strengthening at which the system behaves on periodically closed trajectory or chaotically.

The chaotic oscillations in system (5.2.4) arising as in the first case for autonomous systems and in the second case with periodic $x_g$ *(t)* have been investigated by many scientists. For example, at $f(x) = x (x^2 - 1) (x^2 - r)$ this operated mechanical system has three provisions of balance (special points), and in system arise both periodic on a limit cycle, and chaotic oscillations. Also chaotic oscillations in system (5.2.4) with piecewise and linear function of feedback have been investigated.

## 5.3. Criteria of chaotic oscillations

There is a number of approaches to determination of criteria of emergence chaotic oscillations or chaos in dynamic systems. Criteria share on two types: on predictive or theoretical, allowing to predict emergence of chaos, and on diagnostic or the experimental, allowing to establish existence or absence chaos.

Predictive criterion for emergence prediction chaotic oscillations (movements) is called such criterion which defines set of the operating parameters (or value of separate parameter), leading to chaos. To predictive criteria first of all the criterion of doubling of the period, criteria of a alternation belong and transitional chaos and also criterion of existence of gomoklinical trajectories and Chirikov's criterion about overlapping of resonances for conservative chaos.

Diagnostic criterion of emergence of chaotic oscillations (movements) the test which by results of measurements is called or allows to define data processing whether was or is concrete system in a condition of chaotic dynamics. To diagnostic to criteria the criteria established by means of physical belong and numerical (machine) experiments at which often use such diagnostic characteristics as Lyapunov's indicators and fractal dimension.

**Diagnostic experimental criteria of chaos**

By numerous experiments it is established that chaotic oscillations arise in many *nonlinear systems* in the wide range values of parameters.

A number of examples of emergence of chaos have been considered in chapter III above in this chapter therefore here we will consider only two experimentally investigated an example of chaos, namely an electric circuit from nonlinear inductance and a particle potentially with two holes or a longitudinal bend beams which are characterized by the equation of Duffing.

*1) The compelled oscillations of nonlinear inductance in an electric circuit.*

The equation has been given in the previous paragraph (5.2.3) investigated by Y. Ueda which is presented in the dimensionless form

$$\ddot{x} + k\dot{x} + x^3 = r \cos t, \qquad (5.3.1)$$

where $x$ is current in inductance, $k$ is resistance of a circuit, $r$ is the compelling tension.

Dynamics of system (5.3.1) is defined by two parameters $k$, $r$ and entry conditions ($x(0)$, $\dot{x}(0)$). At a variation of these two parameters, the set of various periodic, subharmonic and chaotic movements which are given in many works, in particular at F. Moon have been received.

*2)The compelled oscillations of a particle potentially with two holes.*

The compelled movements of a particle between two provisions of balance are potential minima with two holes, are described the equation like Duffing

$$\ddot{x} + \delta\dot{x} - 1/2x(1 - x^2) = r \cos \omega t, \qquad (5.3.2)$$

where δ is dimensionless coefficient of attenuation, $r$ is the compelling force, ω – the compelling frequency, non-dimensional quantity by means of the frequency of own small oscillations of system in one of potential holes.

The equation (5.3.2) can describe the movement of a particle in plasma, defect in a solid body or, to loudspeaker of a longitudinal bend of a beam.

As the diagnostic characteristic at the same time serves Lyapunov's indicator. At the same time, on the plane ($r$, ω) at the set attenuation coefficient δ there are areas of chaotic oscillations of a difficult configuration. At to very big compelling $r \gg 1$r , the dynamic mode in system (5.3.2) it will be close to the mode which was investigated by Ueda in case of (5.3.1).

**Predictive (theoretical) criteria of chaos**

Search of theoretical criteria for definition of at what set conditions or parameters, the considered dynamic system will enter in the chaotic mode, is conducted only for a concrete case separately.

The sequence of bifurcations of doubling of the period can be an example, considered in particular, Feigenbaum for square map.

Though these results have been generalized for a wide class of one-dimensional map by means of the renormgrupps theory, criteria of doubling the period it isn't always carried out for map of higher order. That not less, the scenario of doubling of the period is one of possible ways transition to chaos. In more difficult physical systems, understanding of model the Feigenbaum's type it can be useful to definition of when and why there are chaotic movements.

The main theories of chaos resulting in criteria which are useful to forecasting or diagnostics of chaotic behavior in real systems, include the following:

doubling of the period;

gomoklinical trajectories and horseshoe map;

alternation and transitional chaos;

criteria of overlapping of resonances for conservative chaos;

private theories for tasks with a potential having several holes.

A number of criteria from listed above have been considered above in this to the head, therefore further we will stop on the criteria connected with gomoklinical trajectories and horseshoe map and also with criteria of overlapping of resonances for conservative chaos.

*Gomoklinical trajectories and horseshoe map.*

One of theoretical methods which has led to creation of private criteria of chaotic movements, it is based on search of map of type horseshoe and gomoklinical trajectories in mathematical models dynamic systems. Such mathematical procedure known as method Melnikov's, has resulted in the criteria of chaos like Reynolds number connecting system parameters.

The criterion of a gomoklinical trajectory is mathematical by method of receiving a predictive ratio between dimensionless groups of variables of physical system. This criterion gives necessary, but insufficient condition of emergence of chaos. At the same time, criteria gomoklinical trajectory can also generate necessary and sufficient condition of predictability of behavior of dynamic system. At it, in essence this method allows to define whether the model possesses properties of horseshoe map or transformation of the baker.

In case of horseshoe map the attention concentrates on the set of entry conditions for trajectories filling some sphere in phase space. If the system behaves as map of type horseshoes, this initial volume in phase space under action loudspeakers of system takes the new form: initial sphere it is extended and develops (Fig. 5.3). After many iterations these foldings and stretchings generate fraktally structure, and exact information on entry conditions is lost. For establishment compliances between initial and subsequent conditions of system the increasing accuracy is required. With a final accuracy of problem definition prediction becomes impossible.

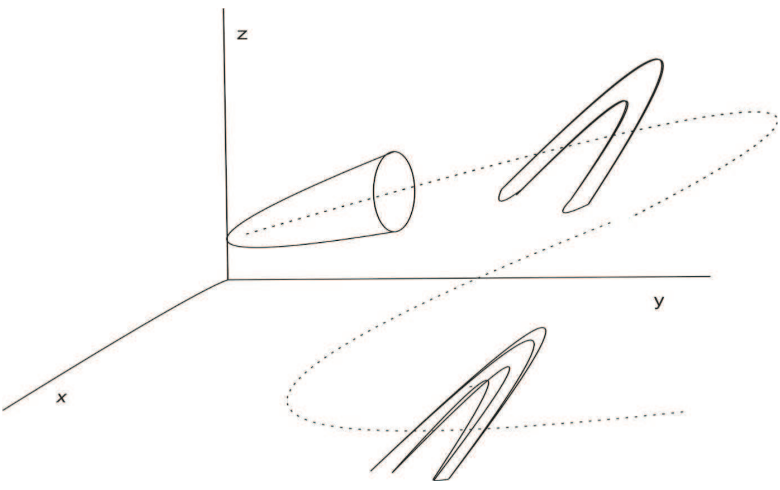



Fig. 5.3.

*Gomoklinical trajectories*. Behavior of nonlinear dynamics and chaos often it is possible to solve, on discrete selection of the movement which is called *Poincare's map*. In Poincare's map of a point, form the sequence of points in $n$ –

measured space, settling down lengthways some continuous curves called by *varieties.*

Gomoklinical trajectories are the sequence the points called by a trajectory.

For example, if it is about a periodic trajectory with the period 3, then the sequence of points serially visits three states on phase planes (Fig. 5.4, a). And the quasiperiodic trajectory corresponds the sequences of the points moving on some closed curve (Fig. 5.6, b).

In dynamics of map special points meet, when passing through which trajectories in one directions move from them, and on another to them. An example thi is a saddle. Such special point has two curves are varieties along which trajectories approach her, and two curves are varieties along which the sequence of points of Poincare is removed from a saddle (Fig. 5.6, c).

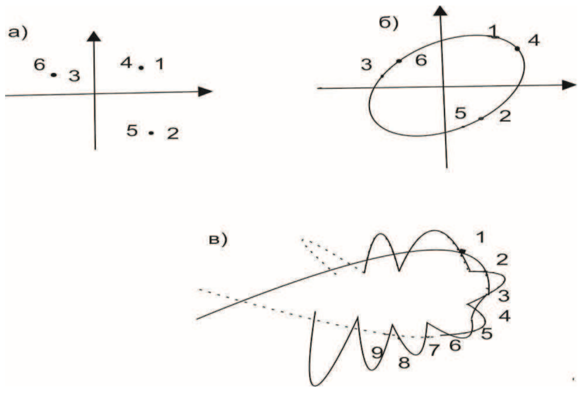



Fig. 5.4.

For understanding of gomoklinical trajectories we will consider dynamics the pendulum oscillating with attenuation under the influence of the compelling force.

Poincare's map synchronized with a frequency compelling forces, has a saddle special point in the vicinity θ = ± *nπ* (*n* is odd), as shown in Fig. 5.5 for a case of the pendulum oscillating under action the compelling force.

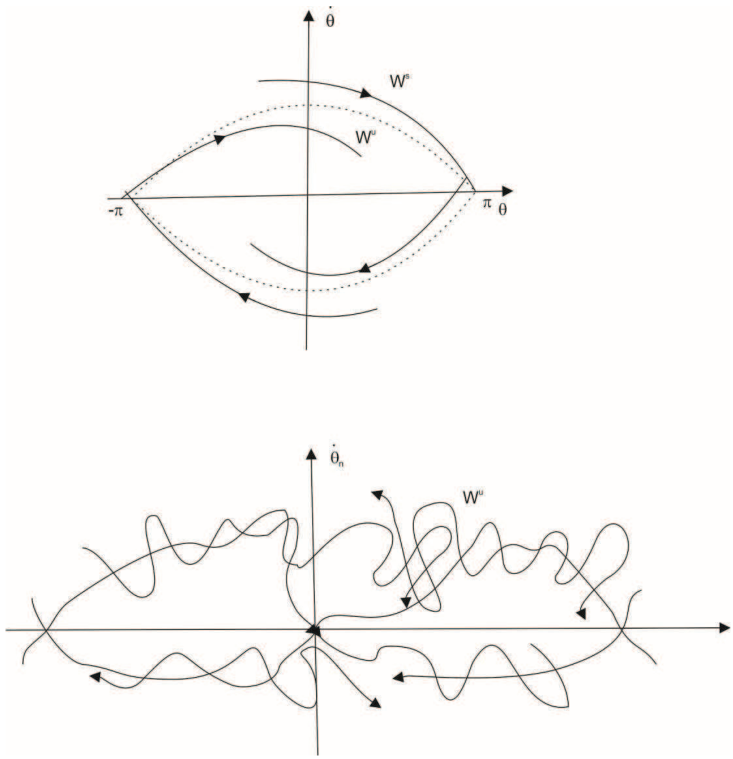



Fig. 5.5.

With rather small amplitude of the compelling force stable and unstable varieties of a saddle don't concern each other. But at increase the compelling force these two varieties are crossed, and it occurs infinite number of times. At the same time points of intersection stable and unstable varieties are called *gomoklinical points.* Point Poincare near one of these points is map on the vicinity of all other points of intersection. The set of such points of Poincare is called *gomoklinical trajectory* (Fig. 5.6, c).

Crossing of stable and unstable varieties at Poincare's map generates in the neighborhood of each gomoklinical the point horseshoe map resulting in unpredictability or sensitive dependence on entry conditions which is distinctive sign of chaos.

Gomoklinical trajectories generate horseshoe maps, so for example, in case of the dissipative system of the area are maped in smaller the areas, and near unstable variety of the area stretch. But as total area has to decrease, the area has to contract quicker, than it stretches. As a result the area near gomoklinical points it develops.

Thus, the dynamic system is considered as transformation of phase space, i.e. volume of the points representing various possible entry conditions, it will be transformed in what over time - that the deformed volume. A regular stream in phase space arises when the transformed volume has smooth outlines. The chaotic stream arises when initial volume stretches, it contracts and develops, as during the *transforming of the baker* or *map like a horseshoe*.

*Chirikov's criterion of overlapping of resonances for conservative chaos*.

Researches of chaotic movements in conservative systems were are begun much more earlier, than in dissipative systems. But cases conservative systems are less widespread and are limited to such areas, as heavenly mechanics, physics of plasma and physicist of accelerators.

As an example in this case the chaos arising is considered at the movement of the jumping ball, it is elastic reflected from horizontal planes. But the differential equations for this case, describe also behavior of the connected nonlinear oscillators and behavior of electrons in electromagnetic field. The equations of blow of gravitating weight about oscillating a surface has an appearance

$$x_{n+1} = x_n + k \sin\varphi_n , \qquad \varphi_{n+1} = \varphi_n + x_{n+1} , \qquad (5.3.3)$$

where $x_n$ is speed before blow, and $\varphi_n$ is time point when occurs blow, rated on the frequency of oscillations of a table, i.e. $\varphi \equiv \omega t$ (mod 2π), $k$ is quantity proportional to amplitude of the oscillating table. As the conservative system (without energy loss), areas is considered entry conditions in phase space ($x, \varphi$) keep the area at repeated iterations of mappings (5.3.3).

Poincare's maps phase trajectories of system (5.3.3) on the plane ($x, \varphi$) at two values $k$ = 0,6 and $k$ = 1,2 are shown in work F. Moon.

So at $k$ = 0,6, points $x$ = 0,2 π correspond to trajectories with the period 1, i.e.

$$x_1 = x_1 + k\, sin\varphi_1 ,$$
$$\varphi_1 = \varphi_1 + x_1 .$$

The solution of this system of the equations has an appearance $\varphi_1 = 0, \pi$; $x_1 = 0$ ($\varphi_1$ and $x_1$ are taken on mod 2π). At the same time the decision close $\varphi = \pi$ is steady at $| 2 - k | < 2$, but close $\varphi = 0{,}2\pi$ it is unstable at $| 2 + k | < 2$ and corresponds to saddle to mapping points.

Close $x = \pi$ the trajectory with the period 2 set by the decision is received systems of the equations

$$x_2 = x_1 + k\, sin\varphi_1 , \qquad \varphi_2 = \varphi_1 + x_2 ,$$
$$x_1 = x_2 + k\, sin\varphi_2 , \qquad \varphi_1 = \varphi_2 + x_1 .$$

And in this case, there are both stable, and unstable points the period 2, it is also claimed that stable points exist at condition $k < 2$.

At $k = 1{,}2$ the movements of the third type are received: near places, where at smaller values of parameter $k$ there were saddles and separatrixes going from a saddle in a saddle, the cloud of points which corresponds is received to *conservative chaos*. At $k < 1$ it is localized in the neighborhood of saddle points. But at $k \approx 1$ wandering trajectory becomes global and "is smeared" on all phase space.

At the same time it is shown that all types of movements can be received, simple the choice of entry conditions (as there is no attenuation, there are no attractors also).

The criterion of global chaos in system (5.3.3) has been offered Soviet the physicist Chirikov who has noticed that at increase in parameter $k$ distance down between the separatrixes connected with periodic the movements of the period 1 and the period 2, decreases. If not intervention of chaos, that separatrixes would be blocked (Fig. 5.6), from here the name of criterion this is *criterion of overlapping.*

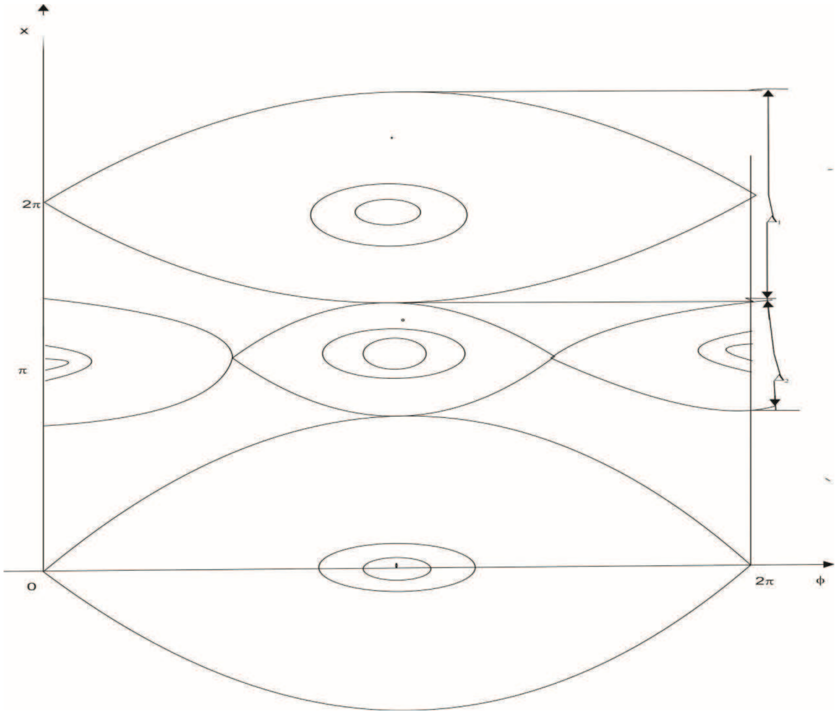



Fig. 5.6.

Standard decomposition (5.3.3) near one of such periodic resonances on small $k$, we will receive that the quantity of the area limited the corresponding separatrix, makes quantity

$$\Delta_1 = 4\, k^{1/2}\,, \quad \Delta_2 = k\,. \tag{5.3.4}$$

In each of decomposition aren't considered influence of other resonances. The condition of overlapping is that $\Delta_1 + \Delta_2 = 2\pi$, or

$$4\, k_c^{1/2} + k_c = 2\pi\,. \tag{5.3.5}$$

From the equation (5.3.5) we find $k_c = 1{,}46$ which is assessment from above critical value $k = k_c$ for the global chaos in number equal $k_c \approx 1{,}0$.

# CHAPTER 6. BASIC PROVISIONS OF THE THEORY AND METHOD TOPOLOGICAL ROUGHNESS OF DYNAMIC SYSTEMS

## 6.1. Topological roughness of dynamic systems

In paragraph 4.1.2. of Chap. 4 definition of roughness of dynamic systems according to Andronov - Potryagin has been given.

It should be noted that a concept of roughness by Andronov - Potryagin the corresponding criteria of roughness determine only qualitative property of roughness of dynamic systems. But this concept allows to enter a quantitative measure of roughness into consideration.

Really, considering conditions of continuous dependence of solutions of systems (4.1.7.) and (4.1.8)

$$\dot{x} = F(x), \qquad \tilde{x}' = F(\tilde{x}) + f(\tilde{x}),$$

from entry conditions and the right parts of these systems, we can claim that for the systems of various topological structures of quantity δ - proximity (4.1.7.) and (4.1.8.), resulting in ε - identity (at fixed small ε > 0), are generally various.

Therefore it is possible to enter the following definitions.

**Definitions 6.1.1.** Rough in the field of *G* system (4.1.7.) is called the *maximal rough* on a set of systems *N* if quantity δ – proximity of systems (4.1.7.) and (4.1.8.), bringing to ε – identity, will be (for everyone ε > 0) it *is maximum*.

**Definition 6.1.2.** Not rough in the field of *G* system (4.1.7.) is called *minimum not rough* on a set of systems *N* if quantity ε-identity at which the roughness condition is still satisfied is (for everyone δ > 0) *minimum.*

**Remark.** A set of *N* in definitions of all dynamic systems formulated above this set which are topological identical each other.

As is well-known from the theory of dynamic systems, necessary and sufficient conditions of roughness are determined by *special trajectories* (*special points, separatrixes, limit cycles, etc.*). The most important special trajectories, in - much *special points* (*SP*) are defining topological structure of system. Possibilities of determination of roughness of dynamic systems on roughness in the neighborhood of *SP* is proved by Grobman-Hartman theorem claiming that in the neighborhood of hyperbolic (rough) *SP* the dynamic system is similar to the linear part.

Conditions of approachibility of the maximum roughness are defined by the theorem given below. But before formulating the theorem, we will enter the following designations:

*M* is a matrix of reduction of a matrix of linear part *A* to diagonal (quasidiagonal) basis with a matrix of, i.e.

$$MQ=AM, \qquad (6.1.1)$$

where $Q = \mathrm{diag}\{\lambda_i,\ i=\overline{1,n}\}$, or $Q = \mathrm{diag}\ \{\lambda_{1,2}^{k} = \delta \pm j\beta,\ k=\overline{1,\ell};\ \ \lambda_i,\ i=\overline{2\ell,n}\ \}$,

and c{M} is conditionality number (usually spectral number) of a matrix *M*. Thus, takes place the following theorem.

**Theorem 6.1.1.** In order that the dynamic system in the neighborhood of a hyperbolic special point ($x_0$) was the maximal rough, and in the vicinity nonhyperbolic is minimum not rough, it is necessary and to have enough

$$M^* = \mathrm{argmin}\ c\ \{M\}. \qquad (6.1.2)$$

*Proof.*

The proof of the theorem 6.1.1 for *n*=2 case is provided in work [163]. We will consider a case of rough system.

*Necessity*. Let the dynamic system set by the equation

$$\dot{x} = F(x), \qquad (6.1.3)$$

where $x \in R^n$, *F* is *n*-a measured vector function, is rough in the area *G*. Then according to criteria of roughness of Andronov - Pontryagin in *SP* of area *G*:

a) det $A \neq 0$, tr $A \neq 0$;

b) if tr $A=0$, then det $A < 0$,

where tr *A* and det *A* are respectively trace and determinant of a matrix *A*.

Let's say system (6.1.3.) rough near a *SP* with a matrix of linear part *A*, i.e. let det $A \neq 0$.

Then, by determination of roughness according to Andronov - Pontryagin, the perturbation system defined in the limited closed subarea $\overline{G}$ of a field of *G*

$$\dot{\tilde{x}} = F(\tilde{x}) + f(\tilde{x}), \tag{6.1.4}$$

will be ε - identical to system (6.1.3).

If now to consider continuous dependence of decisions (6.1.3) and (6.1.4.) from entry conditions and the right parts, for the systems of various topology of quantity δ - proximity (6.1.3) and (6.1.4), bringing to ε - identity (at fixed small ε > 0), are generally various.

But at the same time, the condition of det $A \neq 0$ is equivalent to lack of zero eigenvalue $\lambda_i$ ($i=\overline{1,n}$) matrixes *A*.

At changes of the right parts in system (6.1.4), eigenvalues $\tilde{\lambda}_i$ change in relation to $\lambda_i$ (6.1.3) subjects less, than less quantity of number of conditionality of the matrix of *M* resulting in diagonal (quasidiagonal) basis a matrix of linear part *A* in a *SP* ($x_0$).

$$|\tilde{\lambda}_1 - \lambda_j| \leq c\{M\}\,\|\tilde{A} - A\|, \tag{6.1.5}$$

where $\tilde{\lambda}_I$, $\lambda_j$- $i, j=\overline{1,n}$ are eigenvalues of the perturbation and initial systems in a *SP* with matrixes of a linear part according to $\tilde{A}, A$, $\|\cdot\|$ is any norm of a matrix. Therefore, the conditionality number with {M} estimates quantity δ – proximity of systems (6.1.3) and (6.1.4) at which for fixed ε > 0, roughness conditions are satisfied:

$$(\tilde{D}, (6.1.4.) \overset{\varepsilon}{\equiv} (D, (6.1.3)) \tag{6.1.6}$$

so c{M} is a measure of roughness of initial system (6.1.3).

In that case, there is [163] the maximal rough in a *SP* ($x_0$) system for which $\delta = \delta_{max}$ and $M^* = \text{argmin } c\{M\}$.

*Sufficiency.*

Let the condition be satisfied (6.1.2).

Then, changes $\lambda_i$ matrixes $A$ will be minimum and consequently (6.1.3) will be the maximal rough.

Now we will consider a case of not rough system.

Let the *SP* ($x_0$) of system (6.1.3) be not hyperbolic then, or tr $A \neq 0$, det $A$=0, or tr $A = 0$, det $A > 0$.

By analogy of co a case of rough system (or a hyperbolic point) it is easy to draw a conclusion that not rough systems (6.1.3) can will differ with quantities with the minimum values $\varepsilon_{\min} > 0$ for which it is carried out (6.1.2), i.e. and for not rough systems it is possible to enter a measure of not roughness to which can serve quantity *c{M}*.

Therefore, for minimum not rough system of value $\varepsilon_{\min} > 0$ to which it is carried out (6.1.6), there will be minimum of all set *N* not rough systems, δ – proximity to (6.1.3).

In that case, for minimum not rough system the condition is satisfied (6.1.2) and, on the contrary, if takes place (6.1.2), the system (6.1.3) will be minimum not rough.

Theorem 6.1.1. it is proved.

**Remarks to the theorem 6.1.1.**

1. The possibility of consideration and not hyperbolic *SP* follows from the continuity of function *c{M}*. It is necessary to notice that from the analysis it is known that the concrete decision, for not hyperbolic points strictly speaking doesn't exist (generally *n* of the branching decisions).
2. As appears from definitions 6.1.1 and 6.1.2, and also theorems 6.1.1 exist both minimum rough, and most not rough systems for which with *{M}* →∞. Otherwise, sets of rough and not rough systems form continuous sets in relation to a roughness indicator *c{M}*. At the same time systems for which *c{M}* →∞ there will be systems with a Jordan quasidiagonal form of matrixes of a linear part.
3. Definition 6.1.1 and 6.1.2 and also a condition of the theorem of 6.1.1 invariantly concerning dimension of the considered phase space.

The entered measure from the *c{M}* roughness of dynamic systems allows to solve problems of controlling of roughness of systems in the neighborhood of *SP* of phase space.

Really, let the system is set

$$\dot{x} = \phi(x, u), \tag{6.1.7}$$

where $x \in R^n$, $u \in R^r$ are according to a vector of phase coordinates and control of system, $\phi$ (•) is $n$-measured nonlinear differentiated a vector - function.

Further, let

$$\dot{x} = Ax + B\,u, \tag{6.1.8}$$

is the system of a linear part for (6.1.7) in a *SP* ($x_0$, $u_0$)

$$\Phi\,(x_o, u_o) = 0\ . \tag{6.1.9}$$

In (6.1.8) matrixes $A$ and $B$:

$$A = [\phi\_ixj\^{}'\,(x\_o, u\_o\,)\,], \qquad i, j = \overline{1, n},$$

$$B = \left[\phi'_{iuj}(x_o, u_o)\right], \quad i = \overline{1, n}, \qquad j = \overline{1, r},$$

where $\phi'_{ixj}(x_o, u_o)$, $\phi'_{iuj}(x_o, u_o)$ are respectively private derivatives $\partial\phi_i/\partial x_j$, $\partial\phi_i/\partial \boldsymbol{u}_j$ to a point ($x_0$, $u_o$).

For system (6.1.7) the following theorem is fair.

**Theorem 6.1.2.** In order that in the operated dynamic system (6.1.7) described in phase space of $x \in R^n$ of the vicinity of a *SP* ($x_0$, $u_0$) by means of matrixes of linear part $A$ and $B$, management $\boldsymbol{u} = - Kx$ providing close to of the corresponding *SP* of the closed system with the synthesized matrix $F = A - BK$ the maximum roughness or the minimum not roughness to system (6.1.7) existed, is necessary also enough that conditions of nondegenerate resolvability of the equation of Sylvester for the closed system were satisfied.

*Proof.*

*Necessity.* Let there is $u = - Kx$ translating system (6.1.8) near a *SP* ($x_0$, $u_0$) with a matrix of linear part $A$, in the maximal rough (or minimum not rough) system with a matrix of a linear part in the same point of $F = A - BK$.

Then according to the theorem 6.1.1 it is carried out (6.1.2), so, there is nondegenerate a $M$, such that $MQ = FM$ from where we receive Sylvester's equation

$$MQ - AM = - BH, \qquad (6.1.10)$$

where $H = KM$.

Therefore conditions of nondegenerate resolvability of the equation of Sylvester are satisfied, namely:

1. Controllability of couple ($A, B$);
2. Observability of couple ($Q, H$);
3. Disjointness of ranges $A$ and $Q$;
4. Not degeneracy of BH (at multiple eigenvalues).

*Sufficiency.* Let the condition of nondegenerate resolvability of the equation of Sylvester be satisfied, i.e. there is nondegenerate decision $M$.

Then, synthesizing control

$$u = - Kx, \qquad (6.1.11)$$

where $K = HM^{-1}$, by means of any algorithm (method) of minimization $c\{M\}$ it is possible to reach minimum $c\{M\}$ or $\mathrm{M} = \arg\min \mathrm{c}\,\{\mathrm{M}\}$.

Theorem 6.1.2. it is proved.

Thus, as a result of use of the theorem 6.1.2, the problem of achievement of the maximum roughness comes down to a problem of nonlinear programming of minimization $c\{M\}$, the choice of a matrix of $H$ or ($K$) in Sylvester's equation with the subsequent rationing of the found $M$ matrix (for achievement of uniqueness).

As a result, the algorithm of achievement of the maximum roughness of dynamic systems on set of $SP$ in the area $G$ will be following.

1. $SP$ in the field of $G$ are defined. $P$ will be empty such points.
2. Synthesis of controls of $u_i$, $i=\overline{1,p}$ , such that in $SP$ is provided
$$M_i = \arg\min c\{M_i\}, \qquad \mathrm{i} = \overline{1,\mathrm{p}}\,.$$
3. New coordinates of $SP$ $x_{oj}$ of c numbers $j$ excellent from given, i.e. $j \neq i$ for which are calculated $c\{M_i\}$, $i = \overline{1,p}$ .

4. That control $u^* = u_i = -K_i x$ which provides a minimum gets out $\sum_{i=1}^{p} c\{M_j\}$.
5. The system (6.1.7) with a matrix of a linear part of $F_i = A_i - B_i K_i$ in a *SP* of $x_{0i}$ which corresponds to control u * and will be the maximum rough system in the area *G*.

If in the area *G* except *SP* there are also limit cycles, then the offered method which we will call method of "conditionality of topological roughness" or just method of "topological roughness" allows to extend it and to areas in the neighborhood of limit cycles. We will show it.

Let in the field of G phase spaces there is some limit cycle.

Then if the fundamental matrix of solutions *X (t)* of system is known

$$\dot{x}(t) = \mathrm{F}(x\,(\mathrm{t})), \tag{6.1.12}$$

that on her can be found a matrix of a *monodromy X(T)* of a limit cycle with the *T* period.

If the fundamental matrix of *X (t)* is analytically not found, then the matrix of a monodromy is one of numerical methods, for example by the so-called *firing method.*

By the firing method, for system (6.1.12) are set by some entry conditions

$$x(0) = \text{æ}_{\mathrm{i}}, \qquad i = \overline{1,n}\,, \tag{6.1.13}$$

and certain value of the period of *T*.

Further the system (6.1.12) is integrated from $t$=0 point, to $t=T$ point.

As a result $x_i$ values in a point turn out

$$t = T\colon x_i(T) = \varphi_i(\text{æ}_1, \text{æ}_2 \ldots, \text{æ}_n, T), \quad i = \overline{1,n}. \tag{6.1.14}$$

The matrix of *X (T*) with elements $\partial\varphi_i/\partial\text{æ}_j$, $i,j=\overline{1,n}$ will be a matrix of a monodromy

$$X(T)=[\,\partial\varphi/\,\partial\text{æ}_j],\ i,j=\overline{1,n}. \tag{6.1.15}$$

The matrix of a monodromy *X(T)* unambiguously defines local properties of the vicinity of a limit cycle, in particular, orbital stability is determined by eigenvalues $\mu_j, i = \overline{1,n}$ called by animators of this matrix. If $|\mu_j| < 1$, except one which is always equal +1 then the limit cycle is orbital stable. Otherwise limit cycle orbital unstable.

Now, similar to a case of *SP* it is possible to enter measure $c_T\{M\}$ for assessment of roughness of dynamic systems in the neighborhood of limit cycles as number of conditionality of a matrix of *M(T),* diagonalized (or kvazidiagonalized) a matrix of *X (T),* i.e.

$$c_T\{M\}: \quad M(T)Q(T)=X(T)M(T), \tag{6.1.16}$$

where $Q(T)$=diag $\{\mu_i, \ i = \overline{1,n}\}$,

or $Q(T)$= diag $\{Q_1, \mu_i, i=\overline{3,n}\}$, $Q_1$=[∂,β; ∂,-β].

For limit cycles the theorem is fair.

**Theorem 6.1.3.** In order that in the vicinity orbital - a stable limit cycle dynamic system there was the maximal rough, and in the vicinity orbital - an unstable cycle minimum not rough, is necessary also enough that

$$M(T) = \arg\min c\{M(T)\}. \tag{6.1.17}$$

Theorem 6.1.3. it is proved similar to the theorem 6.1.1.

It is easy to extend a method of "topological roughness" also to piecewise and smooth dynamic systems, considering cumulative roughness on areas of smoothness of system if *SP* aren't on border of these areas. In case of not smooth systems it is possible to expand use of the offered method, using what or the generalized derivative from the rough analysis for definition of a matrix of a linear part.

### 6.2. Application of a method of topological roughness to a research of bifurcations of dynamic systems

We will formulate basic provisions of application of a method of topological roughness to a research of bifurcations in the form of the following problems.

**Theorem 6.2.1.** For not hyperbolic *SP* with one zero valid eigenvalue at changes parametrs $q$ in $q^*$ have point the place

$$c\{M(q^*)\} = \min_{q} c\{ M(q)\}. \qquad (6.2.1)$$

*Proof.* We will apply the theory of perturbations, known from the higher algebra, regarding the provisions based on Gershgorin's theorems.

For definiteness we will carry out the proof for a case of simple eigenvalues. Let the matrix of a linear part of $A_0$ in a special point ($q^*$) have simple eigenvalues and let $\lambda_1(q^*) = 0$.

Further, let $A = A_o + \Delta A = A_o + \varepsilon A_\varepsilon$ is the perturbation matrix close $q = q^*$ point spaces of the parameters $q \epsilon R^\rho$, $\underline{A} = A_o - \varepsilon A_o$ and $\bar{A} = A_o + \varepsilon A_o$, the perturbation matrixes of a near of a point ($q^*$) respectively at the left and to the right of $\lambda_1$ ($q^*$) = 0 on the complex plane of eigenvalues. Here $\varepsilon > 0$ is small quantity.

Then from the theory of perturbations, it is possible to present

$$M_k^{-1} A_k M_k = diag\{\lambda_i^k\} + \varepsilon \begin{bmatrix} \beta_{11}^k/S_1^k & \beta_{12}^k/S_1^k & \beta_{1n}^k/S_1^k \\ \beta_{21}^k/S_2^k & \beta_{22}^k/S_2^k & \beta_{2n}^k/S_2^k \\ \beta_{n1}^k/S_n^k & \beta_{n2}^k/S_n^k & \beta_{nn}^k/S_n^k \end{bmatrix}. \qquad (6.2.2)$$

where $k$ =1,2,3 index carrying the designated quantity respectively to matrixes *A*, $\underline{A}$ и $\bar{A}$; $M_k$ are matrix of diagonalization of $A_k$, $\beta_{ij}^k = (y_i^T)^k A_\varepsilon x_j^k$; $(y_i^T)^k$; $x_i^k$ are respectively rated left and right eigenvector of *i* and *j* of eigenvalues of a matrix $A_k$; $S_i^k = (y_i^T)^k x_i^k$, $i=\overline{1,n}$ are parameters of orientation of vector spaces $(y_i^T)^k$ and (at material $(y_i^T)^k$ and $x_i^k$, $i=\overline{1,n}$ are cosines $x_i^k$ of corners between $(y_i^T)^k$ and $x_i^k$ ).

If to believe that $|g_{\varepsilon ij}| < 1$ is elements $A_\varepsilon$ is that according to the theory of perturbations the center and radius for *i* of a circle of Gershgorin will be equal

$$(\lambda_i^k + \varepsilon\, \beta_{ij}^k/S_i^k), \varepsilon \sum_{j \neq 1} \mid \beta_{ij}^k/S_i^k \mid,$$

or otherwise, for $\lambda_i^k$ of eigenvalue we will have circle radius less than

$$[n\,(n-1)\varepsilon]/|S_i^k|. \qquad (6.2.3)$$

at rather small $\varepsilon > 0$ this circle will be isolated.

We will believe that quantity $\varepsilon$ is identical to all $k$=1,2,3.

Then, indignation quantity $\lambda_i^k$ estimated on (6.2.3) will be will be defined by the quantity $S_i^k$.

We will show that this quantity will be the greatest (from three cases $k$ =1,2,3) for $k$ =1 when $\lambda_i^k$=0.

Really, for $k$ =1 we will have $Ax_1^T = 0$, $y_1^T = 0$, $x_1^T = y_1^T$, $|S_1| = |x_1^T| = 1$.

And for $k$ =2,3, $x_1^T = y_1^T$, since generally $x_1^T \neq y_1^T A^T$, but then $|x_1^{2,3}|<1$.

Therefore, the quality of assessment (6.2.3), so and perturbation of eigenvalue for $k$ =1 $\lambda(q^*) = 0$ will be the smallest that according to the theory of a mazhorization of eigenvalues and vectors means what $c\{M_k\}$ for point $q=q^*$ will be the smallest (the obvious statement that we always have an opportunity to pick up $A_\varepsilon$ is supposed here so that perturbation $\lambda_1$ will be the greatest of all eigenvalues). Theorem 6.2.1. it is proved.

**Remark to the Theorem 6.2.1.**

According to the proof to the theorem 6.2.1. follows that if $\lambda(q_*)=j_0$, i.e. purely imaginary quantity that upon transition through an imaginary axis of eigenvalue $\lambda(q_*)$ (and interfaced $\lambda(q^*) = j_0$) quantity $|S_1|$ generally isn't equal to $q_1$ and consequently, in this case (6.2.1) doesn't take place.

**Theorem 6.2.2.** If in the phase plane a *SP* ($q$ *) such that:

1. Has places that imaginary eigenvalues of a matrix of a linear part;
2. $c\{M(q_*)\} = \min c\{M(q)\}$, it is a *SP* of type difficult

the $q = q_*$ is parameter bifurcation, and at the same time the limit cycle in the neighborhood of this *SP* appears or disappears, i.e. Hopf's bifurcation takes place.

*Proof.* On a condition 1) the *SP* ($q$*) is not hyperbolic and therefore it is a point either like "center", or like "difficult focus" and as at the same time the condition 2) is satisfied about the minimum not roughness.

Vicinities of a *SP*, obviously she can be only like "difficult focus", so bifurcation of emergence (disappearance) of a limit cycle which is called bifurcation Poincare – Andronov – Hopf or as it is accepted in scientific literature Hopf bifurcation takes place.

The simplest example of bifurcation of Hopf is observed for system

$$\dot{x} = -[-q + (x^2+y^2)]x - \omega y, \quad \dot{y} = -[-q + (x^2+y^2)]y + \omega x, \tag{6.2.4}$$

which linear part $[\dot{x}, \dot{y}]^T = A[x, y]^T$,

where matrix $A = \begin{bmatrix} q & -\omega \\ \omega & q \end{bmatrix}$, and eigenvalues $\lambda_{1,2} = q \pm j\omega$.

Upon transition of value through zero value $q$ Hopf bifurcation is observed, eigenvalues quantity crosses an imaginary axis, and $c\{M\} = 1$.

**Theorem 6.2.3.** In order that in the field of *G* multidimensional dynamic systems at value of the parameter $q = q^*$, $q \in R^\rho$, bifurcation of topological structure has arisen, is necessary also enough that:

1) or in the considered area *G* exists not hyperbolic *SP*, or orbital - unstable limit cycles for which takes place:

$$c\{M(q^*)\} = \min_{q} \sum_{i=1}^{\rho} c_i \{M(q^*)\}, \tag{6.2.5}$$

where ρ-quantity of the general points or limit cycles in the area *G*;

2) or in the area *G* dynamic systems hyperbolic points or limit cycles for which the condition is satisfied have:

$$c\{M(q^*)\} = \frac{1}{\rho} \sum_{i=1}^{\rho} c_1 \{M(q^*)\} \to \infty. \tag{6.2.6}$$

*Proof.*

*Necessity.* As is well-known bifurcation means that upon transition through point $q = q^*$ there is a spasmodic change of a picture of nature of movements of phase space in area *G*. Therefore if the case 2) takes place, then it is also a sufficient condition of bifurcation since at the same time there is a break-up of the operator of the vector field (a phase stream) $\dot{x} = F(x)$. That case corresponds to transition of two multiple valid eigenvalues in complex interfaced and vice versa (a Jordan form of a matrix of linear part *A* not diagonal).

If this case isn't carried out, then is necessary performance of a case 1) and we will have not hyperbolic *SP* or orbital unstable limit cycles upon transition through $q=q^*$, such that have to change jump phase streams in *G*.

Let in *G* is available what - or not hyperbolic *SP*. Then in this point, or one of eigenvalues purely imaginary $j_o$ or is equal to zero.

In that case, as it is proved according to theorems 6.2.1 and 6.2.2 in point $q = q^*$ the condition is satisfied

$$C\{M_i(q^*)\} = \min_{q} c\{M_i(q)\},$$

and in the neighborhood of this *SP* occurs what - or local bifurcation. And to area *G* a condition 1) it is rather obvious, occurs or local bifurcation if in some *SP* this condition, or global bifurcation in all area *G* is satisfied if the condition 1) is satisfied in all *SP* of this area.

When performing a condition 2) theorems, it is obvious in the neighborhood of $q=q^*$ there is a global bifurcation if in *G* there is some set of *SP*, or local bifurcation if in *G* only one *SP* and happens change of character of a *SP*, or "saddle" in "knot" and vice versa, or from stable in unstable "focus" and vice versa.

The theorem is proved.

Concerning separatrixes of "saddle" (the third condition of criterion of roughness on Andronov - Pontryagin) the theorem is fair.

**Theorem 6.2.4.** Existence of a separatrix from "saddle" in "saddle", requires also enough that in such couple of *SP* the condition was satisfied

$$c\{M_1\}=c\{M_2\}. \qquad (6.2.7)$$

*Proof.*

*Necessity.* Let for what - or two *SP* like "saddle" there is a separatrix from a saddle in a saddle.

Then, at $n=2$ (on the phase plane) for the first *SP*:

$$\dot{y}/\dot{x} = k = F_2(x, kx)/F_1(x, kx) = f_1(k),$$

or $f_1(k)-k=0$ the equations of a separatrix, where $k$-slope of a separatrix in a *SP*: $F_2'(x, kx)/F_1'(x, kx) = f_2(k)$, $f_2(k) - k = 0$.

Under the assumption: $f_1(k) = f_2(k)$, or

$$F_2(x, kx)/F_1 x, kx) = F_2'(x, kx)/F_1'(x, kx) = f(k). \qquad (6.2.8)$$

For the functions $F_1, F_2, F_1', F_2'$ a general view, the ratio (6.2.8) at any $x$ is possible only if $F_1 = \partial F_1',\ F_2 = \partial F_2',$ where $\partial$ - a constant unequal to zero, i.e. $A_1 = \partial A_2$, therefore $c\{M_1\} = c\{M_2\}$.

*Sufficiency.*

If it is carried out (6.2.7), then $A_1 = \partial A_2,$ or $F_1 = \partial F_1',\ F_2 = \partial F_2',$ is obvious from where we receive (6.2.8).

The theorem is proved.

**Remark.** Theorem 6.2.4. it is proved for $n$=2 case. At $n \geq 3$ it is possible to arrive similarly, consistently considering sections of phase space the planes parallel to the planes of system of coordinates.

The theorems proved in this section allow to use roughness indicators $c(c_T)$ for definition of bifurcations in dynamic systems definition of matrixes of a linear part (or monodromy matrixes) and calculation of $c(c_T)$ in *SP* or on limit cycles.

# CHAPTER 7. APPLICATIONS OF THE THEORY AND METHOD OF TOPOLOGICAL ROUGHNESS TO ROUGHNESS RESEARCHES, BIFURCATIONS AND CHAOS OF SYNERGETIC SYSTEMS

In Chap. 6 bases of the theory and a method of the "topological roughness" allowing to investigate roughness and bifurcations of dynamic systems are stated.

Applications of the developed theory and method of "topological roughness" to a research of roughness, bifurcations and chaos of synergetic systems of various physical nature are presented in this chapter. At the same time some equations and provisions of researches can have repetitions from a number of the previous chapters of the real work, but for integrity of statement of results of this chapter the author has allowed these repetitions.

## System (strange attractor) of Lorenz

It is known that for the first time H. Poincare's opening (1892) that in some mechanical systems described by the determined equations there can be chaotic oscillations was confirmed by the meteorologist E. Lorenz who in 1963 year has offered and investigated mathematical model of thermal convection in the atmosphere. This work Lorenz has opened one of the first examples of the determined chaos in dissipative systems which in his honor carries the name of an *attractor (a strange attractor) of Lorenz.*

In a dimensionless form of the equation of Lorenz take a form

$$\dot{x} = \sigma(y-x), \qquad \dot{y} = px - y - xz, \qquad \dot{z} = xy - \beta z, \tag{7.1}$$

where $x, y, z \in R$ are variable conditions of system, $x$ is proportional to amplitude of speed of the movement, and variables $y, z$ reflect distribution of temperature in a convective ring, σ, ρ are the positive parameters connected with Prandtl's and Rayleigh's numbers, $\beta > 0$ is the parameter characterizing system geometry.

We will take as a basis at a research of the equation and Lorenz's attractor of value of parameters σ and ρ, used by him in work, i.e. $\sigma = 10$, $\beta = 8/3$, and ρ we will vary parameter.

Then the system (7.1) will take a form

$$\dot{x} = 10(y-x), \quad \dot{y} = \rho x - y - xz, \quad \dot{z} = xy - (8/3)z, \tag{7.2}$$

Researches of system (7.2) with use of a measure of roughness $c$ have confirmed the main bifurcations of this system described in literature and answering to conditions of the criteria given in Chap. 6.

Really (fig. 7.1), at $\rho \to 0$, $c(\rho) \to 2.6$, further in process of increase ρ value $c(\rho)$ for existing the only $SP_1(0,0,0)$ types stable "knot" (eigenvalues of a matrix of a linear part $\lambda_{1,2} = -5.5 \pm [20.25 + 10\rho]^{1/2}$; $\lambda_3 = -8/3$ will decrease, having reached value 2.11 at $\rho = r_1 = 1.0$. At $\rho > r_1 = 1.0$ in system (7.2) appear two symmetric relatively $SP_1$ special points $SP_2$ ($a = [(8/3)(\rho-1)]^{1/2}$, $a$, ρ-1) and $SP_3$ ($-a$, $-a$, ρ-1) type stable "focuses", eigenvalues of matrixes of a linear part in which satisfy to the characteristic equation,

$$\lambda^3 + \frac{41}{3}\lambda^2 + \frac{8}{3}(10+\rho)\lambda + \frac{160}{3}(\rho-1) = 0, \tag{7.3}$$

and an average value$\bar{c} = \frac{1}{3}\sum_1^3 c_1(\rho)$ it will be equal to 2.11. The special point of $SP_1$ at $\rho > r_1$ turns in $SP$ type "saddle-knot".

Further, at $\rho \to r_f = 1.345$ the value $\bar{c} \to \infty$ and in system occurs bifurcation of change of type $SP_2$ and $SP_3$, namely: the last turn in $SP$ type stables "focuses".

At $\rho = r_2 = 13.926$ indicator $c$ reach the local minimum equal 1.372, and in system (7.2) there is metastable chaos when attractors $SP_2$ and $SP_3$ of global turn into local attractors with some areas of an attraction (Fig. 7.1). Further, at $\rho = r_4 \approx 24.74$, $\bar{c} = 1.389$ occurs Poincare-Andronov-Hopf (Hopf) bifurcation when eigenvalues $\lambda_{2,3}$ in $SP_{2,3}$ become purely imaginary, equal ± j 9,624. The value of parameter $\rho = r_4$ draws great attention of researchers with the fact that at $\rho > r_4$ in system (7.2) arises the interesting phenomenon called a strange attractor of Lorenz.

In limited area of space $R^3$ around unstable "saddle-fokus" $SP_{2,3}$ arise the chaotic oscillations covering a saddle point of $SP_1$.

Other points and types of bifurcations specified in literature ($r_3$, $r_\infty$, $r_a$, $r_c$ and etc.) so far by means of this measure aren't found and demand an additional

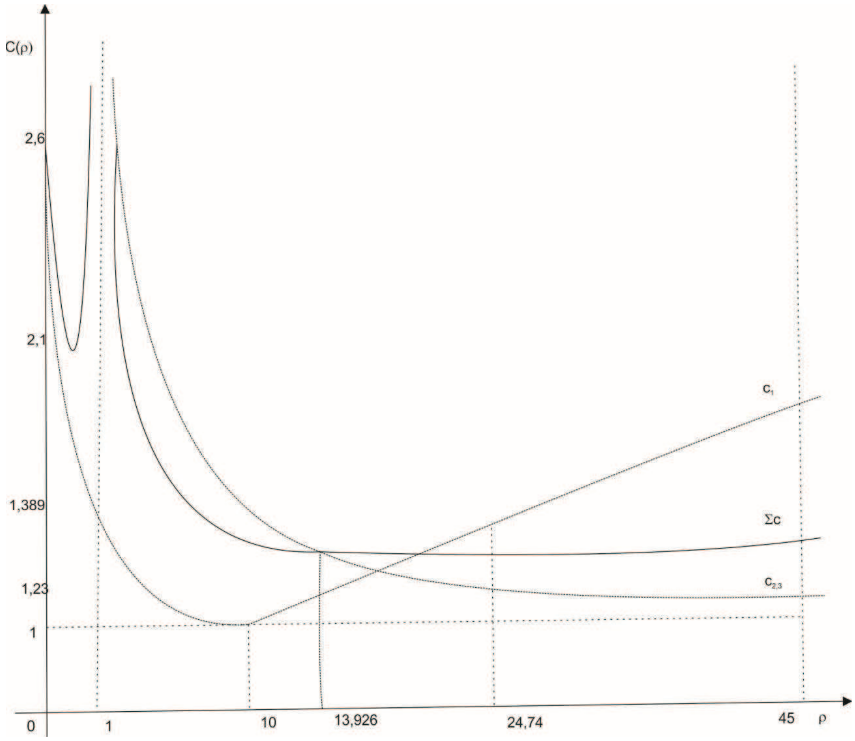


Fig. 7.1. The schedule of dependence $c(\rho)$ for Lorenz's system

research, it is possible with attraction of any nonlinear estimates. But at the same time one more point deserving interest is found namely: a point $\rho = r_m = 45$ where indicators $c(\rho)$ in points $SP_{2,3}$ reach the absolute minimum equal 1.23. In this point it is necessary to expect greatest "stability" of a strange attractor of Lorenz, i.e. small indignations in system (7.2) lead to the minimum changes of area of existence of a strange attractor.

**Rössler's system.**

Rössler's system represents model of chaotic dynamics of the chemical reactions proceeding in some capacity with hashing and is described by the equations:

$$\dot{x} = -y - z, \quad \dot{y} = x + 0{,}2\,y, \quad \dot{z} = 0{,}2 + z\,(x - \mu), \qquad (7.4)$$

where $\mu$ is the varied parameter.

It is known that bifurcations in this system happen through consecutive doubling of the period of a cycle. For calculations of a matrix of a monodromy of a cycle the method known under the name "firing method" is used.

Initial data are set: $T$ period = 1,0, $x$ =-1, $y$ =-1, $z$ = 1,2, an interval of values for $\mu$ from 0 to 11.

The received dependence of an indicator of roughness of $c\{M\}$ on parameter $\mu$ is presented in Fig. 7.2.

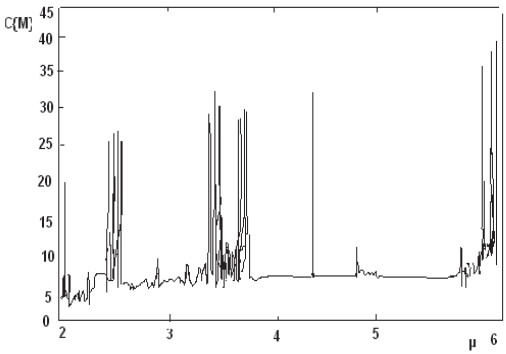



Fig. 7.2. The schedule of dependence of $c\{M\} = f(\mu)$ for Rössler's system

From the schedule in Fig. 7.2 it is visible that the received result on the basis of a method of topological roughness completely corresponds to the results given in literature.

**Belousov-Zhabotinsky system**

This system is described by the equations

$$\dot{x} = k_1 ay + k_2 ax - k_3 xy - 2k_4 x^2,$$

$$\dot{y} = -k_1 ay - k_3 xy + 1/2\, f k_5 bz, \qquad (7.5)$$

$$\dot{z} = 2k_2ax - k_5bz,$$

where $k_1 = 1.28$; $k_2 = 8.0$; $k_3 = 8.0 \bullet 10^5$; $k_4 = 2.0 \bullet 10^3$; $k_5 = 1.0$; $a = 0.06$; $b$=0.020; $0.5 < f < 2.4$ .

Belousov-Zhabotinsky system is a chemical reaction where there are oscillations of concentration of substances and represents catalytic oxidation of $CH_2(COOH)_2$ malon acid. Reaction happens in water solution at the simple shift of the following reagents:

$$[H^+] = 2.0 \text{ mol}; [CH_2(COOH)_2] = 0.28 \text{ mol};$$

$$[BrO_3^-] = 6.3 \cdot 10^{-2} \text{ mol}; [Ce^{4+}] = 2.0 \cdot 10^{-3} \text{ mol}.$$

Reaction is observed on change of coloring of the solution caused by changes of concentration of $Ce^{4+}$ from colourless to yellow.

In system (7.5) depending on $f$ two or three special points ($SP$), one of which the beginning of coordinates. Special points of $SP_i$ $(x_0, y_0, z_0)$ are defined by ratios:

$$x_0 = [6 \cdot 10^{-5} (1-f) - 0.48 \cdot 10^{-7}] \pm \{ [6 \cdot 10^{-5}(1+f) - 0.48 \cdot 10^{-7}]^2$$

$$- 0.1152 \cdot 10^{-10}(1+f) \}^{1/2},$$

$$y_0 = 0.48 \cdot f \cdot x_0 / (0.078 + 8 \cdot 10^5 x_0), \tag{7.6}$$

$$z_0 = 0.48\, x_0 .$$

Results of researches of system (7.6) with use of an indicator of roughness of $c\{M\}$ are shown in Fig. 7.3.

In Belousov-Zhabotinsky reaction various oscillations, including chaotic are found. The last occur at $0.9208 < f < 1.0808$, at $f = 0.9208$, $f = 1.0808.0.9208$ bifurcations. At the same time, the maximum roughness of oscillations is observed at $f \approx 2.0$.

$c\{M\}$

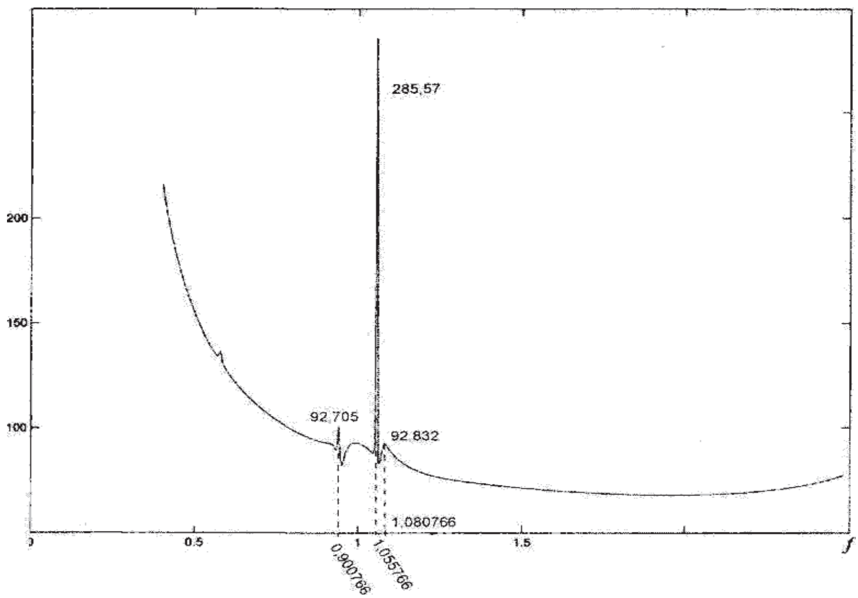



Fig. 7.3. Dependence of an indicator of roughness of $c\{M\}$ on parameter $f$ of the system Belousov – Zhabotinsky

**Predator - prey system**

For the first time this system has been considered by the Italian mathematician of V. Volterra. In a two-dimensional case this system is described by the equations

$$\dot{x} = \alpha x - \beta xy, \qquad \dot{y} = \kappa\beta xy - my, \tag{7.7}$$

where $x, y$ are the number of populations according to the preys and predators, $\alpha,\beta$ are the Malthusian and trophic constant preys showing respectively the growth rate of number of the preys in the absence of predators and the speed of consumption of the preys one predator, $\kappa$ is efficiency of processing of biomass of the prey in biomass of a predator, $m$ is mortality rate of a predator.

An example is reviewed:

$$\dot{x} = -3x + 4x \ - 0.5xy - x \ , \quad \dot{y} = -2.1y + xy. \tag{7.8}$$

In this system four special points: $SP_1$ (0,0); $SP_2$ (1.0, 0); $SP_3$ (3.0, 0); and $SP_4$ (2.1, 1.98). Matrixes of linear parts *SP* are respectively equal in these

$A_1 = [\ -3, 0; 0, -2.1\ ]^T$, $A_2 = [\ 2, -0.5; 0, 0.9\ ]^T$, $A_3 = [\ -6, -1.5; 0, 0.9\ ]^T$,

$A_4 = [\ -0.42, -1.05; 1.98, 0]^T$.

Eigenvalues and types of special points:

$SP_1$: $\lambda_1 = -3$, $\lambda_2 = -2.1$ is "stable knot"; $SP_2$: $\lambda_1 = -1.1$, $\lambda_2 = 2$ is "saddle"; $SP_3$: $\lambda_1 = -6$, $\lambda_2 = 0.9$ is "saddle"; $SP_4$: $\lambda_{1,2} = -0.21 \pm j\ 1.43$, is "stable focus".

We find: $M_1 = [1, 0; 1, 0]^T$, $M_2 = [1, 0.159; 0, 0.987]^T$, $M_3 = [1, 0.2124; 0, -0.977]^T$, $M_4 = [0.389, 0.737; 0.924, -0.676]^T$.

Values $c\{M_i\}$, $i$ =1,2,3,4: $c\{M_1\}$ =1.0; $c\{M_2\}$ =1.174; $c\{M_3\}$ =1.241; $c\{M_4\}$ =1.421.

On a total score $\frac{1}{4}\sum C\{M_i\} = 1.21$ it is visible that this ecological system is rather rough, close to the most rough system when $1/i \sum C\{Mi\} = 1$.

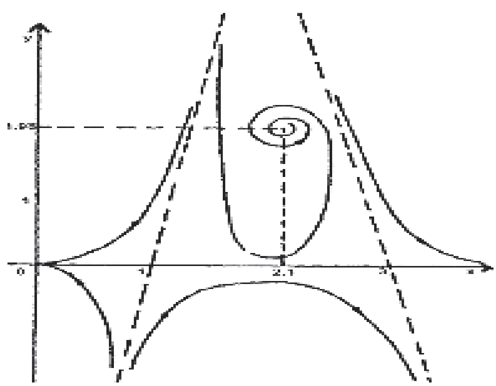



Fig. 7.4. Phase portrait of the predator-prey system

**System (circuit) Chua**

It is known that the Chua system represents an electronic circuit with one nonlinear element which is capable to generate various, in particular, chaotic oscillations.

The Chua system is described by the equations:

$$\dot{x} = p\,(y - f(x))\,, \quad \dot{y} = x - y + z\,, \quad \dot{z} = -\,q\,y\,, \tag{7.9}$$

where $f(x) = M_1 x + 0.5\,(M_1 - M_0)\,(|\,x + 1\,| - |\,x - 1\,|)$.

At $p = 9$, $q = 14.3$, $M_1 = -\,6/7$, $M_0 = 5/7$, in system (7.9) are observed chaotic oscillations.

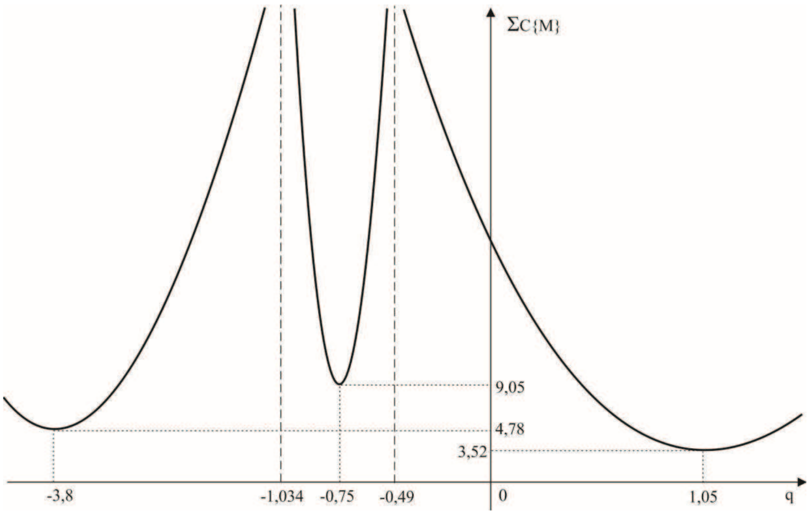



Fig. 7.5. Dependence of *C{M}* on parameter *q* in the Chua system

In this case three special points (*SP*): $SP_1$ (0,0,0); $SP_{2,3}$ (±11/6, 0, 11/6).

Are established by researches that the chaotic movements are found also at values *q*: $-\,1.034 < q < -\,0.49$ , and at $q = -\,3.8$ and $q = 1.05$ the maximum roughness of movements in system (7.9) is observed that are shown in Fig. 7.5.

**Hopf's bifurcation**

This bifurcation is called sometimes Poincare-Andronov-Hopf bifurcation on names of the first researchers of this type of bifurcations. This bifurcation is bifurcation of emergence (disappearance) of a limit cycle in synergetic system.

The simplest example of bifurcation of Hopf is observed for two-dimensional system:

$$\dot{x} = -[-q + (x + y)]x - \omega y, \quad \dot{y} = -[-q + (x + y)]y + \omega x, \tag{7.10}$$

which linear part $[\dot{x}, \dot{y}]^T = A\,[x, y]^T$, where $A = [q, -\omega; \omega, q]^T$, and eigenvalues $\lambda_{1,2} = q \pm j\,\omega$.

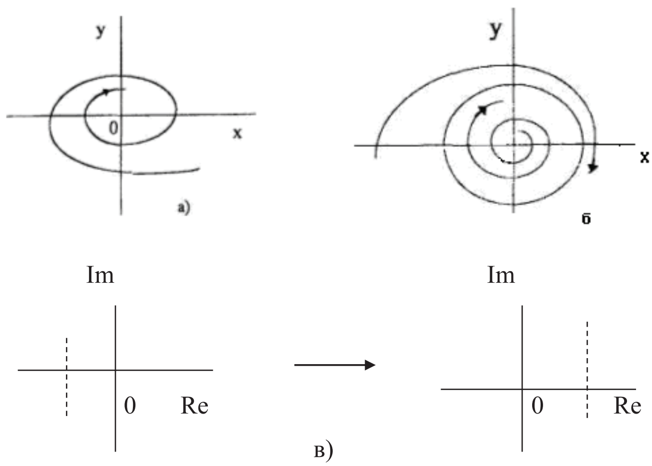



Fig. 7.6. Hopf's bifurcation in system (7.10)

Upon transition of value $q$ through zero value $q = 0$, Hopf's bifurcation is observed (see Fig. 7.6). At the same time, eigenvalues cross an imaginary axis, and quantity C {M} = 1.

More difficult example of bifurcation of Hopf is observed in the three-dimensional system called by the Lengford's system:

$$\begin{aligned} \dot{x} &= (2a - 1)x - y + xz, \\ \dot{y} &= x + (2a - 1)y + yz, \\ \dot{z} &= -az - (x^2 + y^2 + z^2). \end{aligned} \tag{7.11}$$

In this system modeling turbulence in liquid there is Hopf's bifurcation at $a_0 = ½$ and $T_0 = 2\pi$.

The system has two special points of $SP_1$ (0,0,0) and $SP_2$ (0,0,- $a$).

Matrixes of a linear part in special points of $SP_1$ and $SP_2$ respectively

$$A_1 = \begin{bmatrix} 2a-1 & -1 & 0 \\ 1 & 2a-1 & 0 \\ 0 & 0 & -a \end{bmatrix}, \qquad A_2 = \begin{bmatrix} a-1 & -1 & 0 \\ 1 & a-1 & 0 \\ 0 & 0 & a \end{bmatrix},$$

and eigenvalues are equal $\lambda_1 = -a$, $\lambda_{2,3} = 2a - 1 \pm i$ and $\lambda_1 = a$, $\lambda_{2,3} = a - 1 \pm i$.

Quantity $c\{\acute{M}\} = 1$, and upon transition of value $a = ½$ in system (7.11) occurs Hopf's bifurcation.

**Rikitake's dynamo**

The model of a dynamo Rikitake is one of the known models of researches of a magnetic dipole of Earth. In the known works of the researchers brought in the list of references to this work analytical researches of this model have been conducted. Possibilities of observation of a number of the phenomena of a real magneto hydrodynamic dynamo of Earth, such as changes of polarity of a dipole of Earth, oscillation of quantity of a dipole, quasi frequency of change of a dipole are shown.

The model of a dynamo Rikitake is described by the following system of the equations

$$\begin{aligned} \dot{x} &= -\mu x + zy, \\ \dot{y} &= -\alpha x - \mu y + xz, \\ \dot{z} &= 1 - xy, \end{aligned} \tag{7.12}$$

where $\alpha$, $\mu$ are positive parameters, $\alpha = const = \mu\,(k^2 - k^{-2})$, $k$ – coefficient. The case when $k = 2$, $\alpha = 3{,}75\ \mu$ is investigated.

In this case, the system (7.12) in phase space has two special points like "unstable focus".

$SP_1$: ($x_0 = 2, y_0 = 0{,}5, z_0 = 4\mu$),

$SP_2$: ($x_0 =-2, y_0 = - 0{,}5, z_0 = 4\mu$).

Eigenvalues in special points:

$\lambda_1 = -2\mu$, $\lambda_{2,3} = \pm\, i\, 2{,}0615$.

Calculation of $c\{M\}$ at various values $\mu$ has found out that at $\mu = 1{,}1$ and $\mu \to 0$, $c\{M\} = 1$ that confirms conclusions about emergence at value $\mu = 1{,}1$ periodic movements in system (7.12), i.e. there is Hopf's bifurcation.

**Henon's map (Enon's) or horseshoe map**

In the researches Henon is shown that the properties similar to Lorenz's system, the simple discrete map of the plane determined by the equations possesses:

$$x_{n+1} = y_{n+1} - ax_n^2\,, \qquad y_{n+1} = bx_n\,, \qquad (7.13)$$

where $a$ and $b$ are map parameters.

Numerical experiments are made at $a = 1{,}4$; $b = 0{,}3$ (Fig. 7.7).
At this map pulling, compression and folding are made, which after a large number of iterations of map lead to fractal structure.

Depending on the initial point, the sequences of points received by map iterations or go to infinity, or aspire to an attractor.

The attractor to which the map point aspires presents the work of one-dimensional variety on a Cantor set, i.e. has fractal structure. It should be noted that generally the map $T$ transferring some secant $S$ surface to itself (map of each point of $A \to$ transfers $T(A)$ to a point on $S$), is *Poincare's map*.

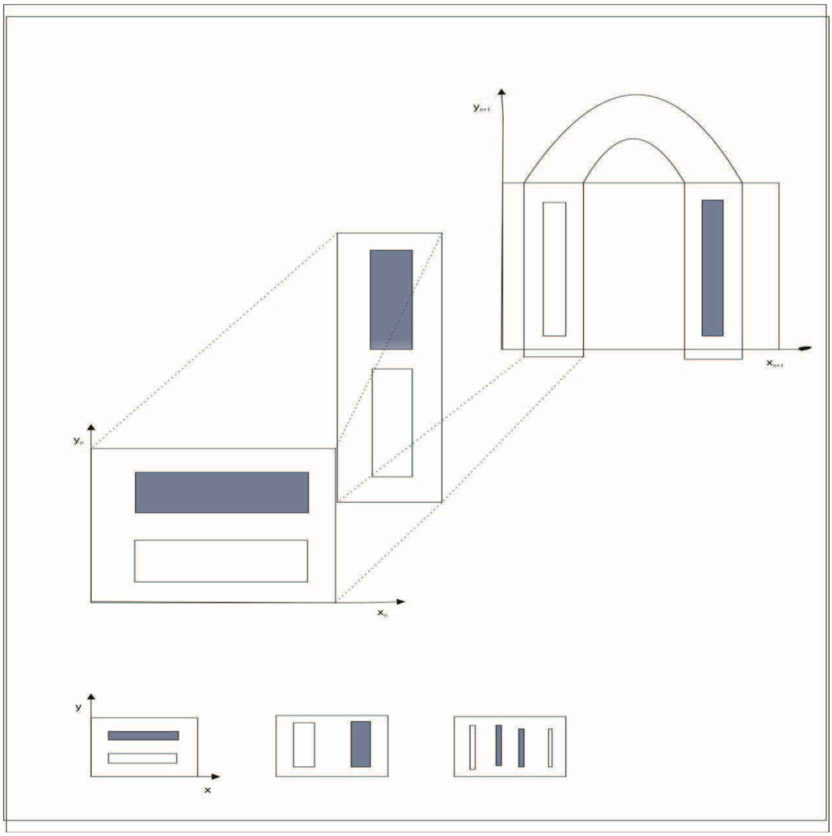



Fig. 7.7. Horseshoe map

Researches of the Henon's map are conducted at $b = 0{,}3$; $a$ is var.

Bifurcation values of parameter $a$:

$a_0 = -\ (1 - b)2\ /4 = -\ 0{,}1225$;

$a_1 = 3(1 - b)2\ /4 = 0{,}3675$;

$a_2 \approx 1{,}06$; $a_3 \approx 1{,}55$.

At point $a < a_0$, *or* $a > a_3$ always go to infinity, at these $a$ the attractor doesn't exist.

At $a_0 < a < a_1$ the attractor is a stable invariant point. When $a > a_1$ an attractor is a periodic set from $p$ of points, similar to a limit cycle in Lorenz's system. With growth and the value $p$ grows and strives for infinity at $a_2 \approx 1{,}06$. At $a_2 < a < a_3$ the attractor is difficult, but not "strange" (not chaotic).

The research by method of topological roughness give the following results:

At $a < a_0$ there are no special points.

At $a = a_0$ only one special point (*SP*):

$x_0 = 1/(1-b) \approx 1{,}43$ , $y_0 = b/(1-b) \approx 0{,}43$.

At $a_0 < a < a_1$ one stable *SP*, and another is unstable *SP*.

State matrix *SP*:

$$A_0 = [-2ax_0, \ \ 1; b, \ \ 0]^T ,$$

Coordinates *SP*:

$$x_0 = 1/2a\,[-0{,}7 \pm \sqrt{(0{,}49 + 4a)}] , \quad y_0 = bx_0 .$$

Eigenvalues: $\lambda_{1,2} = ax_0 \pm \sqrt{(a^2 x_0^2 + b)}$.

The schedule of dependence of $\sum c\{M\} = ½ [\sum c_i^2\{M\}]^{1/2}$, $i = 1,2$ from parameter $a$ is provided by $i$ on fig. 7.8.

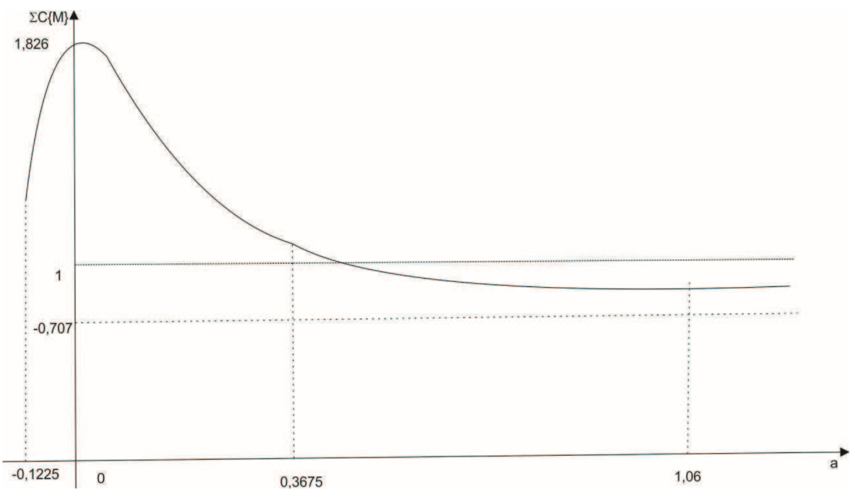



Fig. 7.8. Schedule of dependence of $\sum C\,\{M\} = f(a)$ of the horseshoe map

Researches show that local at least $c\{M\} = 1$ or $\sum c\{M\} = 0{,}707$ isn't achievable at final $a$, and it will be coordinated with the known results of Henon, etc.

**Models of economic systems**

Provisions of the theory and possibility of a method of "topological roughness" are successfully approved on various models of synergetic economy, such as: advanced Kaldor's model; Keynes's model; Solow's model; model like Schumpeter.

Here, we will consider two models of economic systems: the advanced Kaldor's model and model like Schumpeter.

The advanced Kaldor's model characterizes business cycles and it is represented the following system of the equations:

$$\dot{x} = \alpha\,[R(x, z) + I(x, y) - x], \quad \dot{y} = I(x, y) - I_0\ , \qquad (7.14)$$

where $x, y, z$ are respectively variables of national income, capital and welfare; $R(x, z)$, $I(x, y)$ are according to function of expenses on consumption and the volumes of investment; $I_0$ is "replacement" of investments; $\alpha$ is coefficient of adaptation of a cycle (establishment speed).

Function of expenses on consumption:

$$R(x, z) = r(z)x + S(z), \qquad (7.15)$$

and function of savings:

$$T(x, z) = x - R(x, z). \qquad (7.16)$$

Function of investments of $I(x, y)$ has an appearance of the logistic function shown in Fig. 7.9.

Equilibrium state of system (7.14) satisfy to ratios:

$$T(x, z) = I(x, y), \;\; I(x, y) = I_0\,. \qquad (7.17)$$

From Fig. 7.9 obviously that in system (7.14) either one, or three points of balance (special points ($SP$)) are possible.

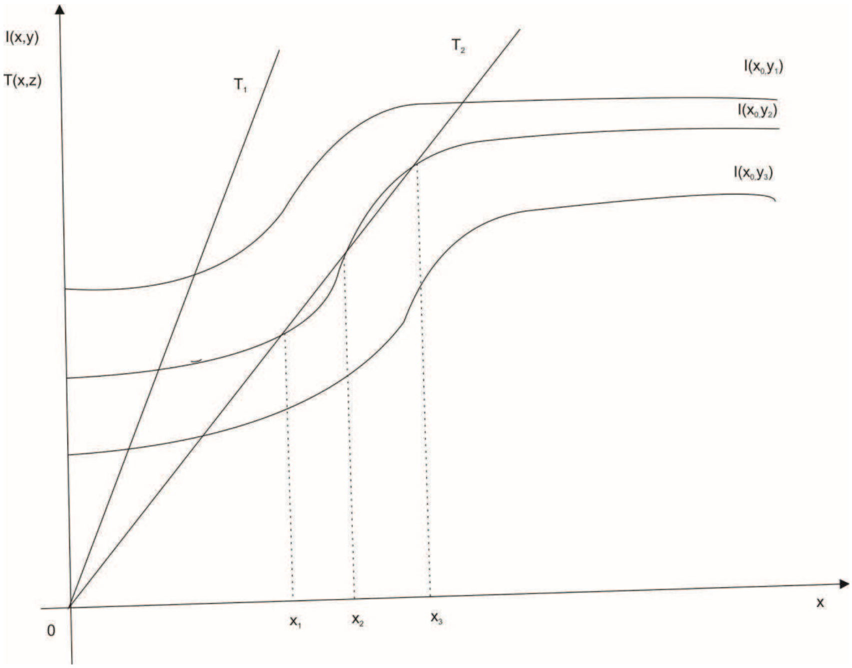



Fig. 7.9. Points of balances of system (7.14)

Believing *R (x, z)* = const, *I (x, y)* = β and in view of functions *I (x, y), T (x, z)* the corresponding fig. 7.9, we will have the following matrixes of linear part (7.14) in vicinities *SP*:

$$A_{1,3} = [-1, -1; 0, -1]^T, \quad A_2 = [\beta - 1, -1; \beta, -1]^T. \qquad (7.18)$$

Respectively in special points ($x_1$, $x_3$) we will have "stable knot", and in a special point ($x_2$):

at $0 < \beta < 2$ is "stable focus";

at $2 < \beta < 4$ is "unstable focus";

at $\beta > 4$ is "unstable knot".

At value $\beta = 2$ in system (7.14) there is a bifurcation of the birth of a limit cycle (Hopf).

The characteristic equation in a special point ($x_2$):

$$\lambda^2 - \lambda(\beta - 2) + 1 = 0,$$

and the indicator of roughness of $c\{M\}$ will be equal to $c\{M\} = \min_\beta c\{M\} \approx 2{,}62$.

Possible stable equilibrium state (special points) have the form of accident (bifurcations) like "assembly".

**Model like Schumpeter**. In the considered model identical to private and state industrial production, the behavior of the investor (and the innovator) and also their strategy in the conditions of the competition, macroeconomics, tolerant to influences, and investments, is investigated by "the induced demand". On this model the non equilibrium movements of economic systems of the industries of the countries and regions are investigated.

Changes of strategy of investments from expansionary character to ratsionalizatsionny and back cause industrial fluctuations. In search of exclusive profits innovators and businessmen pioneers act in the direction opposite to the cyclic movement of investment strategy.

Thus, the model like Schumpeter described by the system of the equations is considered:

$$\dot{x} = sh(y + kx) - xch(y + kx) = P(x, y, k), \qquad (7.19)$$
$$\dot{y} = -\mu[a_0\, sh(8x) + (y - a_1)ch(8x)] = Q(x, y, \gamma),$$

where $x, y$ are respectively variables of the index of configurations of investors and "alternator" - the switch of preferences of the investor between investments of expansionary and ratsionalizatsionny types; $k$ is the coordinator's parameter reflecting intensity of interaction of individual investors; $sh(\cdot)$, $ch(\cdot)$ are functions of a hyperbolic sine and a cosine; $\mu = M/\delta$ is relative parameter, $M$ is the parameter of strategic flexibility of investors concerning change of strategy from expansionary to ratsionalizatsionny and back, $\delta$ is the parameter of temporary scale defining real time of $t = \tau/2\delta$ where $\tau$ is time variable in system (7.19); $\gamma$ is the parameter of speed of a tendency to turn of strategy; $a_1$ is strategy influence parameter is positive at expansionary and negative at ratsionalizatsionny, the choice; $a_0$ is amplitude of the strategic choice.

Conditions of provisions of balance (special points, *SP*):

$$P(x_0, y_0, k) = 0, \quad Q(x_0, y_0, \gamma) = 0,$$

define either one, or three, or five special points which correspond to specifically set parameters of system (7.19).

It is established that in system (7.19) Hopf's bifurcations with emergence of limit cycles are possible.

In particular, at values of parameters: $a_0 = 0{,}5$; $a_1 = 0$; $\gamma = 4{,}0$; $\mu = 0{,}5$, we have the only special point at the beginning of coordinates $x_0 = y_0 = 0$.

The matrix of a yakobian has an appearance:

$$A_0 = [k - 1,\ 1;\ -1,\ -0{,}5]^T .$$

Characteristic equation:

$$\lambda^2 + \lambda(1{,}5 - k) - 0{,}5k + 1{,}5 = 0,$$

and eigenvalues:

$$\lambda_{1,2} = (k - 1{,}5) / 2 \pm 1/2\sqrt{[(k\text{-}1{,}5)^2 + 2k - 6]}.$$

If as the operating parameter to accept parameter *k*, then at values $k = 1{,}5$; $k = 2{,}5$; $k = 3$ there are bifurcations in system (7.19). At the same time, at $k = 1{,}5$ bifurcation of transition from "stable focus" to "unstable focus" is observed. At $k = 2{,}5$ there is a bifurcation of change of topology of space (*x, y*), *SP*(0,0) changes from "unstable focus" on "unstable knot", and at $k \geq 3$ there is a transition to "saddle".

Results of a research of roughness of system (7.19) near (0,0) are shown in Fig. 7.10.

Apparently from the drawing, at $k = 0{,}5$ system will be the most rough with $c\{M\} = 1$, or otherwise, the modelled economic system will be at the same time with the best stability.

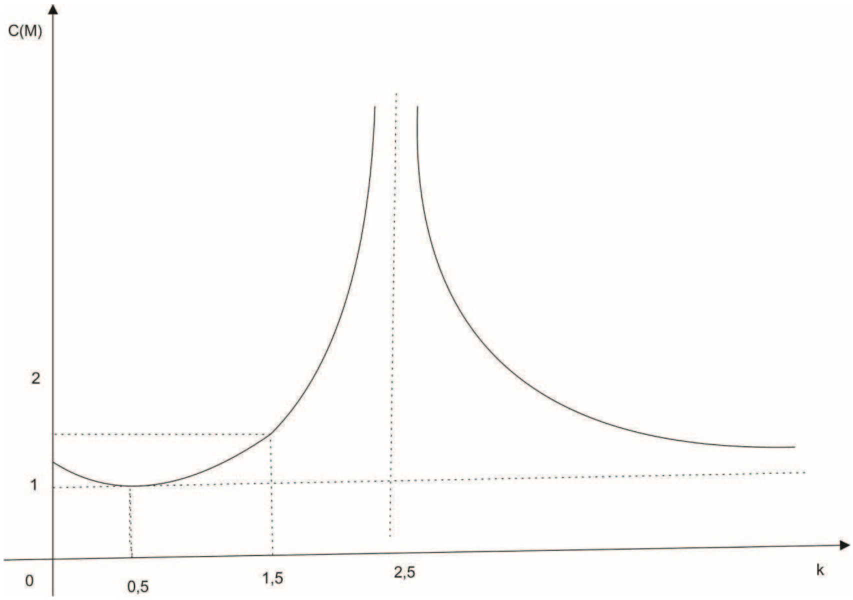



Fig. 7.10. Schedule of dependence of *C {M} = f(k)*

## CONCLUSION

The theory of topological roughness of systems considered in this monograph which applications for a research of synergetic systems of various physical nature are provided in Chap. 7, assumes a formally certain of mathematical model of the studied systems. In case of poorly formalizable and non-formalizable mathematical models of systems, offer application of a method of analogy of set-theoretic topology and abstract approach to researches of such systems.

### Approach bases

Let is set some set of $M_1$ with which is connected other set of $M_2$ of the relation of which is defined by some *morphism* →, i.e. the ratio takes place

$$M_1 \rightarrow M_2 \ , \tag{1}$$

it that

$$F(M_1) = M_2 \ , \tag{2}$$

where $F$ is the *functor* serving as map between sets.

**Definition 1.** The ratio (1) defines some space of sets $\{M\}$ in which the topology of this space of $T$ is defined.

**Definition 2.** μ spaces $\{M\}$ we will call special varieties special points, special lines and multidimensional varieties in this space where certain special (singular) gaps in the ratio (2), in sense of topology of $T$ are possible.

**Definition 3.** We will call *perturbation* of a set of $M$ a set of $F(M)$, it that the $M + F(M)$ forms the *perturbated set* in space $\{M\}$.

**Definition 4.** The metrics δ for perturbations and a metrics ε for the perturbated sets is entered.

**Definition 5.** We will call topology of space $\{M\}$ near some special variety μ rough if at small perturbation δ sets of $M$, the perturbated set of $M + F(M)$ is will cause a stir from a set of $M$ no more than on some small ε.

When determining entered above we can use all basic provisions of the theory of topological roughness of dynamic systems stated in Chap. 6 of this work i.e. consider questions of the maximum roughness and minimum not roughness, etc.

The approach offered here can be used for such poorly formalizable and non-formalizable systems as information systems, social and political systems. At the same time, obviously main difficulty of a research of such systems will consist on definition of the corresponding sets of $M$, functors of $F$, special varieties $\mu$ and also introductions of metrics $\delta$ and $\varepsilon$ spaces $\{M\}$ that is problems of prospect.

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